Money-Metrics in Applied Welfare Analysis: 
A Saddlepoint Rehabilitation*

M. Ali Khan† and Edward E. Schlee‡

18 April 2019

Abstract

Blackorby-Donaldson prove that the money-metric of McKenzie-Samuelson is not generally concave in consumption, and thereby see the sum of money-metrics to be a flawed welfare measure. We present a rehabilitation articulated through Uzawa’s saddle-point theorem of concave programming. Allowing for non-ordered preferences and indivisible commodities, we (i) prove that any competitive equilibrium allocation maximizes the money-metric sum, and thereby furnishes a welfare index conceptually on par with Pareto optimality for price-supportable allocations; and (ii) connect the results to Radner’s local welfare measure and the behavioral welfare proposals of Bernheim-Rangel.

Key words and phrases: Money-metric, saddlepoint inequalities, first fundamental welfare theorem, incomplete preferences, intransitive preferences, non-standard decision theories, behavioral welfare economics, benefit function, distance function, cost-benefit in the small

JEL Classification Numbers: D110, C61, D610.

*We have benefited substantially from the encouragement and comments of Beth Allen, Robert Becker, Gabriel Carroll, Luciano de Castro, Eddie Dekel, Tsogbadral Galaabaatar, Koichi Hamada, Peter Hammond, Michael Jerison, Kevin Reffett, V. Kerry Smith, John Weymark, Nicholas Yannelis, and Itzhak Zilcha. A first draft of this work was written during Khan’s visit to the Department of Economics at Arizona State University (ASU), March 1-6, 2017, and a second draft during Schlee’s visit to the Department of Economics at Ryerson University as Distinguished Lecturer, May 1-5, 2017. Successive drafts were worked on during Khan’s visit to ASU, October 28-30, 2017, and Schlee’s visit to JHU, March 12-15, 2018, and presented at Ryerson, May 2017, the PET Conference, Paris, July 2017, the SERC Conference Singapore August 2017, the ERMAS Conference, Cluj, Romania, August 2017, ASU, September 2017, Midwest Economic Theory Meetings, Dallas, October 2017, the 2018 EWGET Conference, in Paris, July 2018, at Stanford University, September 26, 2018 and the 2019 SWET Conference, Santa Barbara. We gratefully acknowledge the hospitality of these institutions.

†Department of Economics, The Johns Hopkins University. E-mail akhan@jhu.edu
‡Department of Economics, Arizona State University. E-mail edward.schlee@asu.edu
## Contents

1 Introduction 

2 Preliminaries: Notational and Conceptual  
   2.1 Money-metrics: The Background  
   2.2 Example: Money-metrics and Quasi-linear Utility  
   2.3 The Non-concavity of the Money-metric  
   2.4 Saddlepoints and Uzawa’s Theorems  

3 Money-metrics, Saddlepoints and Concave Programming  

4 Competitive Allocations and the Money-metric Sum  

5 Inequality and the Bosmans-Decancq-Ooghe Rehabilitation  

6 Money-metric Relations: Benefit and Distance functions  

7 Money-metrics and Cost-Benefit Analysis in the Small  
   7.1 Radner’s Local Welfare Measure  
   7.2 The Money-metric Sum and Radner’s Measure  
   7.3 Other Welfare Measures in the Small  

8 Concluding Remarks on Future Work  

9 Appendix  
   9.1 Omitted proofs  
   9.2 On the Differentiability of a Money-metric  
   9.3 A Representation Theorem  

3 Money-metrics, Saddlepoints and Concave Programming  

12 Competitive Allocations and the Money-metric Sum  

21 Inequality and the Bosmans-Decancq-Ooghe Rehabilitation  

23 Money-metric Relations: Benefit and Distance functions  

25 Money-metrics and Cost-Benefit Analysis in the Small  

30 Concluding Remarks on Future Work  

30 Appendix  

30 Omitted proofs  

31 On the Differentiability of a Money-metric  

34 A Representation Theorem  

1 Introduction  

7 Preliminaries: Notational and Conceptual  

12 Money-metrics, Saddlepoints and Concave Programming  

15 Competitive Allocations and the Money-metric Sum  

21 Inequality and the Bosmans-Decancq-Ooghe Rehabilitation  

23 Money-metric Relations: Benefit and Distance functions  

25 Money-metrics and Cost-Benefit Analysis in the Small  

30 Concluding Remarks on Future Work  

30 Appendix  

30 Omitted proofs  

31 On the Differentiability of a Money-metric  

34 A Representation Theorem
1 Introduction

The *money-metric* of McKenzie (1957) and Samuelson (1974) gives the smallest income or wealth at fixed prices that leaves a consumer at least as well off as with a given consumption bundle. A startling theorem in an influential paper of Blackorby and Donaldson (1988) asserts that the money-metric cannot be a concave function of consumption except for the knife-edge case of preferences that generate straight-line Engel curves.\(^1\) It is well-understood by a theorem of Debreu and Koopmans (1982) that the quasi-concavity of a sum of functions implies that all but one of the individual functions be concave. The authors draw out what they see to be the negative implications of these two theorems for using the *sum* of money-metrics in welfare economics.

Quasiconcavity [of a social evaluation function on allocations] ensures that social judgments provide goods for everyone rather than giving them exclusively to the few. Samuelson called this requirement “the foundation for the economics of the good society.” Use of money metrics ... may result in a social-evaluation function ... that is not quasiconcave (p. 121). \(\S\)ince many plausible preference orderings yield nonconcave money metrics for all reference prices, we must conclude that social welfare analysis based on money metrics is flawed, despite the fact that money metrics provide exact indices of individual households’ well-being. (p. 129).\(^2\)

If the sum of money metrics is not quasiconcave, then of course the implied social preferences over allocations are not convex. Gorman (1959) had already put forward the objection that non-convexity of social preferences over allocations could lead to discontinuity in optimal allocations as a function of the economy’s parameters (echoed in Blackorby and Donaldson (1988, p. 129)); and Samuelson (1974) himself had openly expressed his skepticism of the use of the money-metric sum in welfare economics, writing:

Since money can be added across people, those obsessed by Pareto-optimality in welfare economics as against interpersonal equity may feel tempted to add money-metric utilities across people and think that there is ethical warrant for maximizing the resulting sum. That would be an illogical perversion....(p. 1266).\(^3\)

More recently, Decancq, Fleurbaey, and Schokkaert (2015, Chapter 2) remark on both the fact and the causes of the withering away of money metrics in welfare economics.

---

\(^1\)Such preferences are called *quasi-homothetic*. Special cases include *homothetic, quasi-linear*, and *Stone-Geary* preferences; see Deaton and Muellbauer (1980, chapter 6), or Mas-Colell, Whinston, and Green (1995, chapter 4) for textbook treatments.


\(^3\)For the record, we agree with Samuelson that there is no *ethical* warrant for maximizing the sum, just as there is no ethical warrant for mere Pareto optimality. We stress that Samuelson’s criticism is based on this very lack of virtue (ethical warrant) and not a positive vice (that the sum is apt to select inequitable allocations); to our knowledge the profession was not aware of the non-concavity until Blackorby and Donaldson (1988).
Money-metric utility ... has a somewhat surprising history. [It] had some impact on the applied welfare economic literature during the eighties [but] lost popularity ... as authors argued that it relied on an arbitrary choice of a reference values and could have nonegalitarian implications. Although it slowly disappeared from the applied welfare economics literature, it was more or less independently developed within the social choice literature in what is called the theory of fair allocation.4

But what are these nonegalitarian implications? At the risk of pedantry, it is not true, as Blackorby-Donaldson write, that quasiconcavity of a social objective implies that its maximizers are interior; nor does non-quasiconcavity imply corner solutions or even inequities. These facts were already pointed out clearly by Gorman (1959, pp. 490-1, Figures 1-2). We give what we believe is the most compelling example of ineegalitarian possibilities in our Example 1 (Section 5). But as we point out there, it depends delicately on the choice of a reference price and on the comparison allocations.5 Our rehabilitation consists of showing that, in a sense we will make precise, the money-metric sum can be rendered neutral with respect to equity considerations, precisely as Samuelson intuited.

Before we develop our argument, we prepare the ground by laying out some antecedent theoretical and empirical literature. An important stream of the last seventy years of applied welfare economics seeks to justify consumers’ surplus calculations in economies that are not quasi-linear.6 The defining simplification of quasi-linearity, and the one that justifies consumer’s surplus calculations, is Marshall’s elusive “constancy of the marginal utility of money”—provided that the the commodity in which the preferences are quasi-linear in is consumed in positive quantities; see Samuelson (1942). Since that good enters utility linearly, its demand alas must be zero for some price-wealth pairs. An attraction of the money-metric lies precisely in its guarantee of the sought-after constancy of the marginal utility of money without the strong assumption of quasi-linearity; and in the effortless way that it handles corner solutions, non-convexities, and even non-ordered preferences.

The money-metric non-concavity has led to a conflicted avoidance as well as its apologetic use. Approaches to the empirical estimation of money-metric welfare meas-

4For the influential papers of the eighties, the authors cite, among others, Deaton and Muellbauer (1980) and King (1983), but for the criticism, highlight only Blackorby and Donaldson’s (1988) influential paper. For the anti-egalitarian implications, see Donaldson (1992, p. 92).

5The only explicitly-calculated example of nonegalitarian implications that we know of is in Fleurbaey and Maniquet (2011, pp. 20-1). Unfortunately, their example is for two identical consumers with quasihomothetic preferences, in which case each consumer’s money metric is concave by Blackorby-Donaldson’s Theorem 2. Their conclusion alas is based on a simple mistake. (Individual 2’s money metric at the unequal allocation is \(1 + \delta\), not \(1 + 2\delta\).) We should say clearly that our rehabilitation is in the spirit of their suggestion just after the example (p. 21) that a judicious choice of a reference price can mitigate nonegalitarian implications of money metrics.

sures fall into three groups: calculation of expenditure functions from parametrically estimated demands; revealed preference; and first- or second-order approximations. Illustrating the Decancq-Fleurbaey-Schokkaert sentiment quoted above, Diewert and Wales (1988, p. 305) explicitly refuse to use the money-metric normalization because of the non-concavity. Banks, Blundell, and Lewbel (1996, p. 1232-3) develop and estimate a quadratic approximation of the money metric to calculate the welfare cost of taxes, but only after acknowledging the Blackorby-Donaldson objections. In their comprehensive project, Deaton and Zaidi (2002) compare the merits of the money-metric sum and a proposal of Blackorby and Donaldson (1987), but write somewhat defensively that, despite the non-concavity, “Our own choice is to stick with money metric utility” (p. 11). They develop aggregate consumption measures for general welfare evaluations based on it. In particular they recommend a first-order approximation to a money-metric utility.

The theoretical response has been either to develop new welfare measures or to rethink the money metric. Blackorby and Donaldson (1987) themselves develop a measure over price-wealth pairs called the welfare ratio that by construction is linear in wealth. Hammond (1994, Section 3.1), after emphasizing the inegalitarian implications of the money-metric sum, goes on to develop a measure based on uniform poll taxes or subsidies. Bosmans, Decancq, and Ooghe (2018) move away from additivity and axiomatize in a social choice framework a (generally) non-additive welfare function of individual money-metrics. Finally, acknowledging the non-concavity, Chambers and Hayashi (2012, p. 811) axiomatize the the aggregate money-metric for social choices involving risk by conceiving of it as the sum of von Neumann-Morgenstern utilities.

With this background, the motivation of this paper can be succinctly underscored. First, we observe that the non-concavity of the money-metric can be circumvented by the remarkable fact that the classical consumer’s constrained maximization problem can be converted into an unconstrained saddlepoint problem with non-concave money metric being the objective function, whether or not it represents the consumer’s preferences. Second, we leverage this observation to prove that any competitive equilibrium allocation maximizes the sum of money-metrics on the set of feasible allocations at the competitive equilibrium prices. This fact puts the sum of money-metrics at the same conceptual

---

7See King (1983) as an example of the first, Varian (1982), Knoblauch (1992) and Blundell, Browning, and Crawford (2003) as examples of the second, and Banks, Blundell, and Lewbel (1996) and Deaton and Zaidi (2002) as examples of the third. Indeed, Varian (2012) explains that his original reason for pursuing revealed preference was to estimate money-metric utilities.

8See also Deaton (1980) and Deaton (2003). The surveys by Blundell, Preston, and Walker (1994, p. 38) and Slesnick (1998, p. 1241) are explicit about non-concavity as an argument against the sum of money metrics.

9We discuss their social-choice rehabilitation of money-metrics in Section 5

10The money metric thus joins other welfare functions that do not necessarily represent consumer preferences but whose sum can be used to identify Pareto optimal allocations. Other examples are the benefit function of Luenberger (1992b) that we discuss in Section 6 and Marshallian consumer’s surplus. Schlee (2013b) develops the last idea both for complete markets and incomplete markets under uncertainty.
level as Pareto optimality of allocations that are supportable as competitive equilibria, providing a foundation for its use by those who, if not in Samuelson’s (1974, p. 1266) phrase “obsessed by Pareto-optimality in welfare economics,” are at least keenly interested in it. Third, we show that a first-order approximation to the money-metric sum can be used to identify small policy changes that are potential Pareto improvements, giving a foundation for empirical work using such approximations. Finally, we conduct our analysis in a setting of preferences that are not necessarily transitive or complete. This fact is surely important in the light of recent efforts to formulate welfare statements about consumers whose choices conform to behavioral models. In particular, we relate the money-metric to the unambiguous choice relations of Bernheim and Rangel (2009) for behavioral welfare economics; see Remark 5 in Section 4.11

Now to a more detailed reader’s guide. After notational and conceptual preliminaries in Section 2, we present the saddlepoint result for individual consumers as Theorem 1 in Section 3. In Section 4, we relate the money-metric to general competitive analysis, and prove that competitive allocations maximize the sum of money-metrics on the set of feasible allocations (Theorem 2). If money-metrics represent consumer preferences, the result thereby yields the first welfare theorem as a simple corollary, an alternative proof that deserves consideration. In Section 5, we present an example illustrating the potential inequitarian implications of maximizing the money-metric sum for an arbitrary price vector, and use the example to explain the relationship between our rehabilitation and the social-choice rehabilitation of Bosmans, Decancq, and Ooghe (2018). In Section 6, we connect our results to two alternative measures: (i) the benefit function originally due to Allais, and comprehensively elucidated in Luenberger (1995), and (ii) the distance function originally due to Malmquist (1953) in the context of consumer theory, and Shepard (1953) in the context of producer theory, and elaborated in Gorman (1970).

In Section 7, we turn to a synthetic overview of cost-benefit in the “small,” and show that the derivative of the money-metric sum equals the derivative of seven other welfare measures in a general equilibrium setting: an equivalence of welfare measures associated with the names of Dupuit, Slutsky, Divisia, Allais-Luenberger, Malmquist-Shepard, and Radner. The equivalence fails for Debreu’s coefficient of resource utilization, and for the measure devised by Hammond (1994) in response to the non-concavity of the money-metric. The concluding Section 8 indicates directions for further analysis stemming from the fact that the saddlepoint conversion of the consumer’s problem to an unconstrained problem allows the direct invocation of lattice-theoretic methods for comparative-statics. The appendix (Section 9) is technical, and can be skipped on a first reading; it gives sufficient conditions for the money-metric to be differentiable, even when preferences are intransitive, a condition we use for local cost-benefit analysis in Section 7.

11Fleurbaey and Schokkaert (2013) extend Bernheim and Rangel (2009) to include equity considerations.
2 Preliminaries: Notational and Conceptual

This four-part section is primarily for the convenience of the reader: we supplement the framework and notation by briefly reviewing pertinent facts about money-metric utilities and concave programming that dictate why the analysis presented in the sequel seemed initially to be so unpromising for applied welfare analysis.

2.1 Money-metrics: The Background

We begin with the basic notation and definitions. There are $L > 1$ commodities, and the consumption set $X \subseteq \mathbb{R}^L_+$ is assumed throughout to be non-empty and closed. The binary relation $\succeq \subseteq X \times X$ describing a consumer’s preferences, is interpreted as weak preference. We write $x \succ y$ if $x \succeq y$ but not $y \succeq x$ (strict preference) and $x \sim y$ if $x \succeq y$ and $y \succeq x$ (indifference). We will impose one of two sets of assumptions on $\succeq$.

**Standard Assumptions.** $\succeq$ is complete, transitive, closed, and locally nonsatiated (LNS).

Note that neither $\succeq$, nor the set over which it is defined, is assumed here to be convex, and, in particular, “discrete” commodities are allowed, as in McKenzie (1957).

Our results also hold for incomplete or intransitive preferences. The next set of assumptions accommodates non-standard decision theories which relax completeness or transitivity, thereby extending our results to behavioral welfare economics. It substitutes for completeness or transitivity *strong convexity* (Shafer (1974, Axiom 3)): for every $z \in X$, if $x \succeq z$, $y \succeq z$, $x \neq y$, and $\alpha \in (0, 1)$, then $\alpha x + (1 - \alpha)y \succ z$.

**Alternative Assumptions.** $\succeq$ is reflexive, strongly convex, and locally nonsatiated; and $\succ$ is open.

For a price-wealth pair $(p, w) \in \mathbb{R}^{L+1}_+$, with $p \neq 0$, the budget set and the demand correspondence are

$$B(p, w) = \{x \in X \mid p \cdot x \leq w\} \quad \text{and} \quad d(p, w) = \{y \in B(p, w) \mid x \succ y \implies x \notin B(p, w)\}.$$
Under the Standard Assumptions, the money-metric $M(\cdot, \cdot)$ is the value function for the following wealth-minimization problem\(^{15}\)

$$M(x, p) = \min_{x' \succsim x} p \cdot x' \text{ with } h(p, x) \equiv \arg\min_{x' \succsim x} p \cdot x'. \quad (1)$$

The money-metric can be routinely redefined for the Alternative Assumptions as\(^{16}\)

$$M(x, p) = \inf_{x' \succsim x} p \cdot x'. \quad (2)$$

If $\succsim$ is closed, then the two definitions agree.\(^{17}\) Since $\succsim$ is reflexive, the constraint set in (2) is nonempty; and since $X$ is a closed subset of $\mathbb{R}^L_+$, the infimum is nonnegative. We shall refer to the function $h(\cdot, \cdot)$ as the (Hicksian) compensated demand to distinguish it from the (Marshallian) demands defined above.

In the classic textbook case of $X = \mathbb{R}^L_+$ and the Standard Assumptions plus strong monotonicity ($x > y$ implies $x \succ y$), the money-metric at price $p$ assigns the wealth level $w$ to the indifference set passing through the point $x = d(p, w)$. Since preferences are stongly monotone, this assignment of numbers to indifference sets represents $\succsim$ on $X$ in this case; see Figure 2 (but ignore for now the points $a$, $b$ and $x'$). In this case the money-metric is related to the familiar expenditure function, defined as $e(p, \bar{u}) = \min_{u(x') \geq \bar{u}} p \cdot x'$, by substituting for $\bar{u}$ the utility level $u(x)$ attained by the commodity bundle $x$. This is to say that $M(x, p) = e(p, u(x))$.

The money metric is related to two other classical welfare measures: the compensating variation and the equivalent variation. The equivalent variation for a move from price-wealth pair $(p^0, w^0)$ to $(p^1, w^1)$ is

$$EV(p^0, (p^1, w^1)) = M(d(p^1, w^1), p^0) - M(d(p^0, w^0), p^0);$$

and the compensating variation for that move is

$$CV(p^0, (p^1, w^1)) = M(d(p^1, w^1), p^1) - M(d(p^0, w^0), p^1).$$

If, as we will do in Section 4, take $p^0$ to be an equilibrium price, the equivalent varia-

\(^{15}\) $M(\cdot, \cdot)$ is the minimum income function in McKenzie (1957), who emphasized it as a function of $p$ for fixed $x$. Samuelson (1974, pp. 1272-1273) clearly ascribed the concept to McKenzie but studied it as a function of $x$ for a fixed $p$. Given his emphasis, he changed the nomenclature of an expenditure or a minimum income function to a money-metric, and it is this change that illuminates the inherent duality of the object and opened up new vistas and applications for it.

\(^{16}\) McKenzie (1958) proposes this definition in the absence of continuity.

\(^{17}\) In the Alternative Assumptions we require that $\succ$ is open and explained the rationale for it in footnote 14. The reason we do not also assume that $\succsim$ is closed is the Theorem in Schmeidler (1971). It asserts that if $X$ is a connected topological space, $\succsim$ is closed and transitive, $\succ$ is open and nontrivial ($x \succ y$ for some $x, y$ in $X$), then $\succsim$ is complete. We want to allow the possibility that $X$ is connected and $\succsim$ is transitive, but not complete.
tion is a special case of the money metric, in that the money metric is defined for all consumption plans \( x \in X \), not just those that are demanded at some price-wealth pair.\(^\text{18}\)

### 2.2 Example: Money-metrics and Quasi-linear Utility

We now calculate the money-metric for the familiar and historically-important special case of quasi-linear preferences defined on two goods, \( X = \mathbb{R}^2_+ \), and represented by the quasi-linear-in-good-2 form \( u(x_1, x_2) = \phi(x_1) + x_2 \), where \( \phi \) is strictly concave and differentiable with \( \phi' > 0 \). The strict concavity of \( \phi \) implies that preferences are strongly convex. For this subsection, we take good 2 to be *numeraire* and set \( p_2 = 1 \).

For \( p_1 > 0 \), define \( f(p_1) \) to be the unique number (if any) that solves the Kuhn-Tucker conditions for the consumer’s problem of choosing \( x_1 \geq 0 \) to maximize \( \phi(x_1) + w - p_1 x_1 \), i.e., by ignoring the non-negativity constraint on \( x_2 \). These conditions are

\[
\phi'(f(p_1)) - p_1 f(p_1) \leq 0 \quad f(p_1) [\phi'(f(p_1)) - p_1 f(p_1)] = 0, \quad \text{and} f(p_1) \geq 0.
\]

If there is no solution to these conditions, set \( f(p_1) = \infty \). The demand for good 1 equals \( f(p_1) \) if \( f(p_1) \leq w/p_1 \) (that is, the constraint \( x_2 \geq 0 \) does not bind), and equals \( w/p_1 \) otherwise. The indirect utility function for the consumer is \( v(p_1, w) = \phi(f(p_1)) - p_1 f(p_1) + w \) if \( w \geq p_1 f(p_1) \) and \( \phi(w/p_1) \) otherwise. Note that the marginal utility of wealth equals 1 if and only if the non-negativity constraint on good 2 *does not bind*. For reference utility \( \bar{u} \) in the range of \( u \), the expenditure function equals \( e(p_1, \bar{u}) = \bar{u} - \phi(f(p_1)) + p_1 f(p_1) \) if \( \bar{u} \geq \phi(f(p_1)) \) and equals \( p_1 \phi^{-1}(\bar{u}) \) otherwise. It follows that

\(^{18}\)With the proviso that \( p^0 \) is taken as an equilibrium price, the equivalent variation is what King (1983) advocates as a welfare measure; he calls it *equivalent income*. 

---

**Figure 1: Monetary-labelled indifference curves**
the money-metric is given by

\[ M(x, p) = \begin{cases} 
  u(x_1, x_2) - \left[ \phi(f(p_1)) - p_1 f(p_1) \right] & \text{if } u(x_1, x_2) \geq \phi(f(p_1)) \\
  p_1 \phi^{-1}(u(x_1, x_2)) & \text{otherwise} 
\end{cases} \]

(3)

In particular, for fixed \( p \), \( M(x, p) = u(x_1, x_2) \) up to an additive constant if and only if the non-negativity constraint on \( x_2 \) in the expenditure minimization problem does not bind at \( p \); otherwise the two functions diverge.\(^{19}\) Of course, since \( x_2 \) enters the representation linearly and \( X = \mathbb{R}^2_+ \), the non-negativity constraint for \( x_2 \) must bind for a region of values for \( (p, \bar{u}) \), namely those for which the compensated demand for good 2 is 0. Applications that hope to preserve the simplicity of quasi-linearity then, often implicitly, restrict parameters to avoid corner solutions for the *numeraire* good. And these parameters from the viewpoint of the consumer are endogenous equilibrium variables from the viewpoint of the economy as a whole.\(^{20}\) A virtue of the money-metric is that it preserves the constancy of the marginal utility of money without requiring positive consumption of any particular good.

### 2.3 The Non-concavity of the Money-metric

In order to help the reader see the non-concavity of a money-metric, we outline an alternative proof of the Blackorby-Donaldson theorem that uses the least-concave representation result of Debreu (1976); see Khan and Schlee (2017). Let \( X = \mathbb{R}^L_+ \), suppose the Standard Assumptions on preferences hold, and that \( \succsim \) is strongly monotone (\( x > y \) implies \( x \succ y \)). In this case the money metric itself is a preference representation; see Weymark (1985a) or Section 9.3 below. For any representation \( u \), write the money-metric as \( M(x, p) = e_u(p, u(x)) \), where \( e_u \) is the expenditure function associated with \( u \).

There are two cases. Either there is no concave representation of preferences, in which case we are done;\(^{21}\) or there is a concave representation \( u \) of preferences. In the second case, the indirect utility function is concave and strictly increasing in \( w \) for each \( p \).\(^{22}\)

Since the expenditure function \( e_u(p, \cdot) \) is the inverse of the indirect utility for fixed \( p \), \( e_u(p, \cdot) \) is convex. So \( M(\cdot, p) \) is a convex transformation of any concave representation;

---

\(^{19}\)For example, if \( \phi(x_1) = \ln x_1 \), then \( M(x, p) = [\ln x_1 + x_2] + \ln p_1 + 1 \) if \( \ln x_1 + x_2 \geq -\ln p_1 \), and \( M(x, p) = p_1 x_1 \exp(x_2) \) otherwise.

\(^{20}\)One workaround is simply to drop the non-negativity constraint on the *numeraire* good altogether and assume its consumption can be any real number, as in Mas-Colell, Whinston, and Green (1995, Chapter 3, p. 44 and Section 10.C).

\(^{21}\)The Finetti-Fenchel-Kannai examples highlight how a convex preference relation may not have a concave representation for precisely such plausible preference orderings. See Kannai (1977) and Kannai (1981) for examples and the antecedent background.

\(^{22}\)If the consumer has expected utility preferences and we interpret \( u \) as a von Neumann-Morgenstern utility, this fact implies that aversion for wealth risk follows from aversion for consumption risk; see Kreps (2013, Proposition 6.16, p. 136).
which is to say that it is less concave than any concave representation.\footnote{For two continuous real-valued functions $f$ and $g$ on a convex set $D \subseteq \mathbb{R}^n$ that represent the same binary relation $\succeq$, $g$ is less concave than $f$ if there is a convex, and strictly increasing, real-valued function $T$ on $\text{Range}(f)$ that is convex with $g = T \circ f$.} By Debreu (1976), there is a least concave representation, $\tilde{u}$; that is, if $u$ is a concave representation, then $u = T \circ \tilde{u}$ for some concave, strictly increasing function $T : \text{Range}(\tilde{u}) \to \mathbb{R}$. So either i) $M(\cdot, p)$ is affinely related to $\tilde{u}$—that is $M(x, p) = a(p) + b(p)\tilde{u}(x)$ for $b(p) > 0$ and hence the money-metric is also a least concave representation; or ii) it is not concave, which completes the argument.

The literature is silent about the region of consumption the money-metric is not concave, but Samuelson (1974, p. 1276) makes an intriguing assertion. He writes that the money-metric $M(x, p^0)$ is "locally concave in all $x$ that are near to $x^0$ [a point demanded at $p^0$] (and which may as well be restricted to $x$'s that are at least as good as $x^0$)." But what he elucidates symbolically is

$$\nabla_x M(x^0, p^0) \cdot (x - x^0) \geq M(x, p^0) - M(x^0, p^0)$$

(4)

for all $x$ in a neighborhood of $x^0$, that is, that $M(\cdot, p^0)$, is concave precisely the demanded point $x^0$.\footnote{By "concave at the point $x^*$" for a differentiable function $f$ we mean that $\nabla f(x) \cdot (y - x) \geq f(y) - f(x)$ for all $y$ in some neighborhood of $x$.} Whereas (4) is certainly true if money metric is differentiable at a demanded point and the demanded point is in the interior of the consumption set, Samuelson’s verbal assertion about concavity in a neighborhood of a demand point remains to this day an open question.\footnote{For what it is worth, it was reflection on Samuelson’s assertions about local concavity of money-metric that lead us to think about what global properties of concave functions, if any, that it preserves, independently of differentiability and interiority assumptions, and thereby to Uzawa’s saddlepoint theorems.}

### 2.4 Saddlepoints and Uzawa’s Theorems

Here we recall for the reader Uzawa’s (1958) saddlepoint theorems. Let $\emptyset \neq Z \subseteq \mathbb{R}^n_+$, and let $f : Z \to \mathbb{R}$ and $g : Z \to \mathbb{R}^m$ be functions, where $n$ and $m$ are positive integers. Consider the constrained optimization problem max$_{z \in Z} f(z)$ subject to $g(z) \leq 0$, and the saddlepoint inequalities asserting the existence of $z^* \geq 0$ and $\lambda^* \geq 0$ such that

$$\mathcal{L}(z, \lambda^*) \leq \mathcal{L}(z^*, \lambda^*) \leq \mathcal{L}(z^*, \lambda)$$

for all $z \in Z$ and $\lambda \in \mathbb{R}^m_+$, and where $\mathcal{L}(z, \lambda) = f(z) - \lambda \cdot g(z)$.

We can now present

**Theorems** (Uzawa (1958)). 1. If $(z^*, \lambda^*)$ satisfies the saddlepoint inequalities, then $z^*$ solves the optimization problem. 2. If $z^*$ solves the optimization problem, $Z$ is convex,
If the saddlepoint inequalities hold, then \( z^* \) maximizes \( f(z) - \lambda^* g(z) \) on \( Z \) ignoring the constraint \( g(z) \leq 0 \). A difficulty in applying Uzawa’s Theorem 1 is to verify that the saddlepoint inequalities hold. The deeper Theorem 2 gives sufficient conditions for the saddlepoint to hold; it applies to the consumer’s problem if preferences are representable by a concave function. There are two well-known impediments for a successful application. First, the value of the multiplier, the marginal utility of money, depends on the parameters of the problem, and thereby complicates its use for comparative statics. Second, it requires a concave representation of preferences, and as already pointed out in the introduction, even if one imposes convexity of preferences, some convex preference relations are not representable by a concave function.\(^{26}\)

### 3 Money-metrics, Saddlepoints and Concave Programming

If \( \succeq \) is transitive, it follows that

\[
x \succeq y \quad \text{implies that} \quad M(x, p) \geq M(y, p)
\]  

(5)

since the constraint set for the income minimization problem (2) at \( x \) is a subset of the constraint set for that problem at \( y \). If \( \succeq \) is complete and transitive, it follows if \( x' \in d(p, w) \), then \( x' \) maximizes \( M(\cdot, p) \) on the budget set \( B(p, w) \). In this section we are after something bolder: that \( x' \) maximizes \( M(x, p) - p \cdot x \) on \( X \) ignoring the budget constraint, even if preferences are non-ordered. Of course if \( M(\cdot, p) \) represents \( \succeq \), then any maximizer \( x' \) of \( M(\cdot, p) \) on \( B(p, w) \) is a demand. But without further restrictions on the consumption set, a money metric does not in general represent preferences even under the Standard Assumptions: in particular, we can have \( x \succ y \), but \( M(x, p) = M(y, p) \).\(^{27}\)

The possibility is depicted in Figure 2. The consumption set equals the vertical bars, preferences are strongly monotone, and \( x^2 \succeq x^1 \succeq x^0 \). At the price \( p = (1/2, 1/2) \), \( M(x^1, p) = M(x^2, p) \). But nothing prevents \( x^2 \succ x^1 \). In that case \( M(\cdot, p) \) does not represent the consumer’s preferences on \( X \), and \( x^1 \notin d(p, p \cdot x^1) \) even though it maximizes \( M(x, p) \) on the budget set \( B(p, w) \). We now introduce a variant of the condition that

\(^{26}\)See the references in Footnote 21. We remind the reader that concavity cannot be relaxed to quasi-concavity in Uzawa’s Theorem 2 even if the constraint function \( g \) is linear. The Cobb-Douglas function \( u(x_1, x_2) = x_1 x_2 \) for the consumer’s problem is a counterexample.

\(^{27}\)Weymark (1985a) is the classic reference for money metrics as a preference representation.
Figure 2: The consumption set equals the vertical bars, so the consumption set is not convex. Assume that preferences are strongly monotone. If \( x^2 \succ x^1 \succeq x^0 \), then \( x = x^1 \) violates the local cheaper point at \( p^0 = (1/2, 1/2) \). The point \( x^2 \) is demanded at \( (p^0, M(p^0, x^1)) \), but there is no point in a small-enough neighborhood of \( x^2 \) that is cheaper at those prices. We have \( M(p^0, x^1) = M(p^0, x^2) \), even though \( x^2 \succ x^1 \). The curves in the Figure are not indifference curves but are meant to be a visual aid.

McKenzie (1957) uses in his derivation of the Slutsky equation.  

**Definition 1.** A point \( x \in X \) satisfies the **local cheaper point condition** at \( p \in \mathbb{R}_+^L \) if, for every \( x' \in d(p, M(p, x)) \) and every open neighborhood \( N \) of \( x' \), there is a point \( x'' \in N \cap X \) such that \( p \cdot x'' < p \cdot x' \).

In Figure 2, if \( x^2 \succ x^1 \succeq x^0 \), then \( x^1 \) violates the local cheaper point condition since \( x^2 \in d(p, p \cdot x^1) \), but there is no point in \( X \) that is cheaper than it at price \( (1/2, 1/2) \). Of course \( x^2 \) itself violates the cheaper point condition as well, but this violation does not pose a problem for us since \( x^2 \) is itself demanded at price \( p \) and income \( p \cdot x^2 \): \( x^2 \in d(p, p \cdot x^2) \). The difficulty for \( x^1 \) is that \( x^1 \notin d(p, p \cdot x^1) \).

The role of the local cheaper point condition is spelled out in the next lemma. With it, compensated and uncompensated demands are equal; and the constraint \( x' \succeq x \) in the wealth minimization problem (2) binds. We prove the Lemma in the Appendix.

**Lemma 1.** Fix \( p >> 0 \). If \( \succeq \) satisfies the Standard Assumptions, and \( x \in X \) satisfies the local cheaper point assumption at \( p \), then

\( (a) \) \( h(p, x) = d(p, M(p, x)) \); and \( (b) \) \( x' \sim x \) for any \( x' \in h(p, x) \).

Of course in the classic case \( X = \mathbb{R}_+^L \) the local cheaper point assumption holds for every \( x \neq 0 \). Again, the point \( x = 0 \) poses no problem for us since it is itself demanded at zero wealth at any price: \( 0 \in d(p, 0) \).

We now turn to our main result for a single consumer. It lays the foundation for our agenda on the role of money metrics in consumer theory.

\[ \text{\textsuperscript{28}} \text{We warn the that the cheaper point assumption is being formulated in a slightly different way than in Walrasian general equilibrium theory where for each price, it is the existence of a cheaper point in the budget set of each consumer that is being asserted.} \]
Theorem 1 (Money-Metric Saddlepoint). Fix \((p,w) > 0\). Define \(\mathcal{L}(x,\lambda) = M(x,p) + \lambda[w - p \cdot x]\). Consider the saddlepoint inequalities for \((x^*,1)\) with \(x^* \in X\):

\[
\mathcal{L}(x,1) \leq \mathcal{L}(x^*,1) \leq \mathcal{L}(x^*,\lambda) \quad \text{for every } x \in X \text{ and } \lambda \geq 0. \tag{6}
\]

(a) If \(x^* \in d(p,w)\) and \(\succeq\) satisfies either (i) the Standard Assumptions or (ii) the Alternative Assumptions, then the saddlepoint inequalities (6) hold.

(b) If \(p >> 0\), the saddlepoint inequalities (6) hold, and if \(x^*\) satisfies the local cheaper-point condition at \(p\), then under the Standard Assumptions on \(\succeq\), \(x^* \in d(p,w)\) for \(w = p \cdot x^*\).

Note that \(M(\cdot,p)\) need not even be quasiconcave. Note too that part (a) holds even if the money metric does not represent preferences at the reference price \(p\), as in the example in Figure 2, or indeed if preferences are neither complete nor transitive, so that there is no representation at all. We leave the proofs of Theorems 1(a) and 2(a) in the text because they are short and simple. The part-(b) proofs are in the Appendix.

Proof of (a): Since \(M(x,p) - p \cdot x \leq 0\) for every \(x \in X\), the difference \(M(x,p) - p \cdot x\) is certainly maximized whenever it equals zero. Let \(x^* \in d(p,w)\) for some \(w \geq 0\). Under either (i) or (ii), \(\succeq\) is reflexive and locally nonsatiated. By local nonsatiation, \(p \cdot x^* = w\). And since \(\succeq\) is reflexive, \(x^*\) is in the constraint set for the wealth minimization problem at \(x = x^*\). In case (i), a familiar argument establishes that if \(y \succeq x^*\), then \(p \cdot y \geq w\): if \(p \cdot y < w\), then by local nonsatiation, there is a \(z \in X\) with \(p \cdot z < w\) and \(z \succ y\); since \(x^* \in d(p,w)\) and \(\succeq\) is complete, \(x^* \succeq z\); by transitivity, \(x^* \succ y\). It follows that \(M(x^*,p) = w\), and so \(M(x^*,p) - p \cdot x^* = 0\). In case (ii) consider any \(z \succeq x^*\) with \(z \neq x^*\). By strong convexity, for any \(\alpha \in (0,1)\), \(\alpha z + (1 - \alpha)x^* \succ x^*\). Since \(x^* \in d(p,w)\), \(p \cdot (\alpha z + (1 - \alpha)x^*) > w\), which after rearranging implies that \(p \cdot z > w\). Again, the infimum of \(p \cdot z\) over all such \(z \succeq x^*\) is at least \(w\), so \(M(x^*,p) = w\) and \(M(x^*,p) - p \cdot x^* = 0\). \(\square\)

We now specialize to the classic case of \(X = \mathbb{R}^L_+\) to illustrate a use of Theorem 1. It asserts that the wealth expansion path at a price \(p\) equals the set of maximizers of \(M(x,p) - p \cdot x\) on \(X\). It has the implication that we can replace the standard utility maximization problem in classical consumer theory—maximize utility subject to a budget constraint—with an unconstrained optimization problem using the money metric, whether or not it represents preferences. It illustrates that for some well-behaved consumption sets, we can dispense with completeness and transitivity to get the conclusion of Theorem 1(b). For Corollary 1, we don’t need the full conclusion of Lemma 1, but simply that \(x^* \in d(p,p \cdot x')\) if and only if \(p \cdot x' = M(x',p)\).\(^{29}\)

\(^{29}\)Absent transitivity, it is well-known that the equality in Lemma 1(a) fails; see Fountain (1981).
Corollary 1. Let \( X = \mathbb{R}_+^L \) and \( p > 0 \). If either (I) the Minimal Assumptions; or (II) the Alternative Assumption hold, then \( x' \in d(p, w) \) for some \( w \geq 0 \) if and only if \( x' \) maximizes \( M(x, p) - p \cdot x \) on \( X \).

The corollary puts consumer theory on the same footing as producer theory: maximizing benefit minus cost on a consumption or production set, at least for the classic case \( X = \mathbb{R}_+^L \). In a sequel to this paper we use this formulation to simplify dramatically the comparative statics of the consumer’s problem. A long-recognized difficulty in applying the lattice-theoretic comparative statics methodology of Topkis (1995) and Milgrom and Shannon (1994) to the consumer’s problem is that different budget sets do not stand in the strong-set-order relation.\(^{30}\) Corollary 1 solves this problem by dispensing with the budget set altogether.

**Proof of Corollary 1:** If \( x' \in d(p, w) \) then, by Theorem 1(a), \( x' \) maximizes \( M(x, p) - p \cdot x \) on \( X \) for either (I) or (II). Now suppose that \( x' \) maximizes \( M(x, p) - p \cdot x \) on \( X = \mathbb{R}_+^L \).

Since \( 0 \in X \) and \( M(0, p) = 0 \), it follows that \( 0 = M(0, p) - p \cdot 0 \leq M(x', p) - p \cdot x' \leq 0 \) which implies that \( M(x', p) = p \cdot x' \). Set \( w = M(x', p) \geq 0 \). We will show that \( y \in B(p, w) \) implies that \( y \not\succ x' \). Let \( y \in B(p, w) \). If \( x' = 0 \), then \( w = 0 \) and the budget set \( B(p, w) \) is a singleton, so the conclusion holds since \( \succsim \) is reflexive. Let \( x' \neq 0 \), which since \( X = \mathbb{R}_+^L \) implies that \( x' > 0 \). If \( p \cdot y < w \), then \( y \not\succ x' \) by the definition of \( M(x', p) = \inf_{z \succeq x'} p \cdot z \) and \( w = M(x', p) \). If \( p \cdot y = w \), then since \( x' \neq 0 \) and \( p >> 0 \), we have \( w > 0 \), which in turn implies that \( y > 0 \). Consider the sequence \( y^n = (1 - \frac{1}{n})y \). Fix \( n \). Since \( p \cdot y^n < w \), it follows that \( y^n \in \{ z \in X \mid z \not\succ x' \} \), which is a closed set under either (I) or (II). We then have \( \lim_{n \to \infty} y^n = y \not\succ x' \), so \( x' \in d(p, w) \).

\[ \square \]

4 Competitive Allocations and the Money-metric Sum

We now extend the framework of Section 2 to \( I \) consumers. This is to say that each consumer \( i \) has a consumption set \( X_i \subseteq \mathbb{R}_+^L \) that is nonempty and closed; and a preference relation \( \succsim_i \subseteq X_i \times X_i \) satisfying either the Standard or Alternative Assumptions. Define \( M_i(x, p) = \inf_{(x', p) \in X_i \times X_i} p \cdot x' \) with \( (p, x) \in \mathbb{R}_+^L \times X_i \). For \( (p, w) \in \mathbb{R}_+^{L+1} \) and \( B(p, w) = \{ x \in X \mid p \cdot x \leq w \} \), we let \( d_i(p, w) = \{ x' \in B(p, w) \mid x \succ_i x' \) implies \( x \not\in B(p, w) \} \).

Let there be \( J \geq 1 \) firms, with firm \( j \) endowed with a production set \( Y_j \subseteq \mathbb{R}_+^L \) satisfying \( 0 \in Y_j \) (possibility of inaction). The aggregate production set is \( Y = \sum_j Y_j \), assumed to be closed. Let

\[ \pi_j(p) = \max_{y_j \in Y_j} p \cdot y_j \]

\(^{30}\)See for example Quah (2007) and Mirman and Ruble (2008).
when it exists. Consumers are endowed with goods and ownership shares in firms. Consumer $i$’s price-dependent wealth is $w_i(p) = p \cdot \omega_i + \sum_j \theta_{ij} \pi_j(p)$, where $\omega_i \in X_i$ is consumer $i$’s endowment of goods and $\theta_{ij}$ is $i$’s ownership share of firm $j$. We assume that $\omega = \sum_i \omega_i > 0$.

**Definition 2.** Let $A = \prod_i X_i \times \prod_j Y_j$. An allocation is a point $(x_1, ..., x_I, y_1, ..., y_J) \in A$. An allocation $(x_1, ..., x_I, y^1, ..., y_J)$ is feasible if

$$\sum_i x_i \leq \sum_i \omega_i + \sum_j y_j.$$ 

We will denote an allocation by $(x, y)$. We denote aggregate consumption by $x = \sum_i x_i$, aggregate production by $y = \sum_j y_j$, and the aggregate endowment by $\omega = \sum \omega_i$.

**Definition 3.** A competitive equilibrium is a feasible allocation $(x^*, y^*)$ and a nonzero price vector $p^* \in \mathbb{R}^L_+$ such that

1. For every $j = 1, ..., J$, $p^* \cdot y^*_j \geq p^* \cdot y_j$ for every $y_j \in Y_j$;
2. For every $i = 1, ..., I$, $x_i \succ_i x_i^*$ implies $x_i \notin B(p^*, w_i(p^*))$.

For each consumer $i$ and $p \in \mathbb{R}^L_+$, let $X_i^{lep}(p) \subseteq X_i$ be the set of consumption plans that satisfy the local cheaper point condition at $p$.

We now show that any competitive equilibrium allocation maximizes the sum of money metric utilities on the set of feasible allocations—despite the non-concavity, and even though the money metric need not represent preferences. We will follow, for example, if, for every good $\ell$, some consumer’s preferences were strictly monotone in that good, or if the production set $Y$ satisfies free disposal.

**Theorem 2.** Let $L_p(x, y, \mu) = \sum_i M_i(x_i, p) + \mu \cdot [\omega + y - x]$. Consider the following saddlepoint inequalities at $(x^*, y^*, p^*) \in A \times \mathbb{R}^L_+$:

$$L_{p^*}(x, y, p^*) \leq L_{p^*}(x^*, y^*, p^*) \leq L_{p^*}(x^*, y^*, \mu)$$

for every $(x, y, \mu) \in A \times \mathbb{R}^L_+$. (7)

(a) If $(x^*, y^*, p^*)$ is a competitive equilibrium, and each consumer’s preferences satisfy either the Standard or the Alternative Assumptions, then the saddlepoint inequalities (7) hold. It follows that $(x^*, y^*)$ maximizes $\sum_i M_i(x_i, p^*)$ on the set of feasible allocations.

---

31 Our results go through under two kinds of market incompleteness. The first is that there simply aren’t markets for some goods, so consumers must consume their endowments of those goods. The second is that every consumer must consume the same physical good across a common set of states, locations, or dates. On the inclusion of public goods and non-priced commodities as parameters of the money-metric, see Hammond (1994), Fleurbaey and Maniquet (2011) and Fleurbaey and Blanchet (2013).
(b) If the saddlepoint inequalities (7) hold at some \( p^* \gg 0 \); if each consumer’s preferences satisfy the Standard Assumptions; and if for every consumer \( i \), \( x_i^* \in X_i^{lep}(p^*) \); then \((x^*,y^*,p^*)\) is a competitive equilibrium for some distribution of endowments and ownership shares.

Proof of (a): Let \((x^*,y^*,p^*)\) be a competitive equilibrium. Since for any \( \mu \in \mathbb{R}^L_+ \), \( \mu \cdot [\omega + y^* - x^*] \geq 0 \) and, by local nonsatiation, \( p^* \cdot [\omega + y^* - x^*] = 0 \), certainly \( L_{p^*}(x^*,y^*,p^*) \leq L_{p^*}(x^*,y^*,\mu) \) for every \( \mu \in \mathbb{R}^L_+ \). Profit-maximization for each firm \( j \) at \( p = p^* \) implies that \( p^* \cdot y^* \geq p^* \cdot y \) for every \( y \in Y \); and for each consumer \( i \), \( M_i(p^*,x_i) - p^* \cdot x_i \leq 0 \) for every \( x_i \), with equality at \( x_i^* \) by Theorem 1(a). So \( L_{p^*}(x,y,p^*) \leq L_{p^*}(x^*,y^*,p^*) = p^* \cdot (\omega + y^*) \). Since the saddlepoint inequalities (7) hold, it follows from Theorem 1 in Uzawa (1958) (part 1. of the Theorem in Section 2.4 here) that the competitive allocation maximizes the sum of money metrics on the set of feasible allocations. \( \square \)

Figure 2 helps illustrate a difference between parts (a) and (b). Suppose it describes a single-consumer economy with endowment equal to \( x^0 \) and that the line \( x_1 + x_2 = \text{const} \) describes the set of final consumption bundles obtainable with the economy’s constant-returns technology. If \( x^2 \succ x^1 \succ x^0 \) then the unique competitive equilibrium allocation is \( x = x^2 \). The plan \( x^2 \) maximizes the consumer’s money metric at \( p = (1/2, 1/2) \). The point \( x^2 \) is not covered by part (b) since it violates the local cheaper point assumption. If however \( x^1 \succ x^2 \succ x^0 \), then the equilibrium allocation is \( x^1 \), which satisfies the local cheaper point assumption and solves the saddlepoint inequalities at \( p = (1/2, 1/2) \). It is also a competitive equilibrium, illustrating part (b).

We conclude this section with a series of remarks.

Remark 1 (The money-metric sum and (in)justice). Whatever allocation one thinks is a just one, if it can supported as a competitive equilibrium, then the money-metric sum is maximized at that allocation (if evaluated at the corresponding equilibrium price). The same can be said of course about any unjust allocation, however defined, supportable as a competitive equilibrium. Theorem 2 simply puts the money-metric sum on the same conceptual level as Pareto Optimal allocations supportable as competitive equilibria. \( \square \)

Remark 2 (The saddlepoint and the wealth of nations). The last sentence of Theorem 2(a) asserts that a competitive equilibrium allocation maximizes the money metric sum on the set of feasible allocations. The saddlepoint inequalities (7) imply something deeper: a competitive allocation maximizes the monetary sum \( \sum_i M_i(x_i,p^*) - p^* \cdot x + p^* \cdot y \) on the set \( \mathcal{A} \) of allocations, feasible or not. The maximized value of this sum equals simply \( p^* \cdot y^* \), competitive equilibrium aggregate profit. \( \square \)
Related to this Remark, the next result is an immediate implication of Corollary 1 that we write out formally for completeness. The point is that we can solve for the allocations demanded at a price ignoring the budget constraint.

**Corollary 2.** Let \( X_i = \mathbb{R}_+^L \) for every consumer \( i \), and suppose that each consumer’s preferences satisfy either the Standard or Alternative Assumptions. Then \( x^* \in (d_1(p,w_1), \ldots, d_I(p,w_I)) \) for some \((w_1, \ldots, w_I) \geq 0\) if and only if \( x^* \) maximizes \( \sum_i M_i(x_i,p) - p \cdot x \) on \( \mathbb{R}_+^L \).

**Remark 3** (First Welfare Theorem). If for each consumer \( i \), \( M_i(\cdot,p^*) \) represents \( \succeq_i \) on \( X_i \), then the sum of money metrics is itself a Bergson-Samuelson welfare function. Theorem 2(a) asserts that any competitive allocation maximizes this sum at the corresponding competitive price. So we get as a corollary to Theorem 2(a) the first welfare theorem. Saddlepoints are a commonplace in the literature on existence and optimality of competitive equilibria when consumers have concave utility representations.\(^{32}\)

Theorem 2 shows that a competitive equilibrium allocation maximizes the sum of money metrics if the reference price is the associated competitive equilibrium price. That includes any other competitive equilibrium allocations, supportable at a possibly-different price. This distinguishes the sum of money metrics from the sum of compensating variations, since generally the sum of compensating variations is positive when moving from one competitive equilibrium to another.\(^{33}\)

**Remark 4** (Evaluation of non-competitive allocations). What can be said about how the sum ranks other allocations? That depends on the underlying preferences. There are three possibilities. In what follows fix a reference price (which might or might not be a competitive equilibrium price).

1. **Each consumer \( i \)’s preferences are represented by \( M_i(\cdot,p) \).** In this case the sum is a Bergson-Samuelson social welfare function (a function from utility profiles to real numbers that is strictly increasing in each consumer’s utility): anything that maximizes the sum will be Pareto optimal; and if the sum is higher at consumption allocation \( x' \) than \( x \), then at least one consumer \( i \) strictly prefers \( x'_i \) to \( x_i \). The second property is sometimes called *Pareto consistency*.

2. **Each consumer’s preferences are complete and transitive, but for some consumer \( i \), \( M_i(\cdot,p) \) does not represent \( i \)’s preferences.** In this case the sum is no longer a

\(^{32}\)Negishi (1960) is the classic reference for the differentiable case and Takayama and El-Hodiri (1968) for the general case.

\(^{33}\)This possibility has become known as the Boadway (1974) paradox. Like the sum of equivalent variations, the sum of money metrics is nonpositive when moving from one competitive allocation to another. This fact was one consideration in King’s (1983) adoption of the money-metric for welfare.
Figure 3: The curve is the boundary of the at-least-as-good-set at $x_i$. Since $p \cdot z_i < M_i(x_i, p)$, point $z_i$ cannot be strictly preferred to $x_i$. If $\succ_i$ is complete, then $x_i \succ z_i$. It follows that, in the notation of Bernheim and Rangel (2009), that $(z_i, x_i) \notin R'$; and if $z_i$ and $x_i$ are comparable under $\succeq_i$, then $x_i P^* z_i$ in their notation.

Bergson-Samuelson social welfare function, and an allocation that maximizes the sum need not be Pareto optimal. (Interpret the straight line in Figure 2 to be the set of attainable allocations.) But the sum is still Pareto consistent: if it rises, then at least one consumer is better off. This follows from our equation (5).

3. Some consumer’s preferences are either incomplete or intransitive. In this case the sum of money metrics need not be Pareto consistent. But the sum can still be used to identify aggregate consumption plans that are not a potential Pareto improvement relative to some allocation $x$. To see this, fix a consumer, a reference price $p$ and a consumption plan $x_i$. If $p \cdot y_i < M_i(p, x_i)$, then $y_i \not\succ x_i$. Of course if $x_i \in d_i(p, p \cdot x_i)$, this is just the statement that $x_i$ is directly strictly revealed preferred to $y_i$. But the money metric reveals more than this, since the point $x_i$ need not be demanded at $p$; see Figure 3. Now consider a consumption allocation $x$ and an aggregate consumption $x'$. If $p \cdot x' < \sum_i M_i(x_i, p)$, then the aggregate consumption plan $x'$ cannot be a potential Pareto improvement over $x$. If $x'_i \succ x_i$ for $i = 1, ..., I - 1$, then $p \cdot x'_i \geq M_i(x_i, p)$ for $i = 1, ..., I - 1$. Sum over $i$ to find $p \cdot \sum_{i=1}^{I-1} x'_i \geq \sum_{i=1}^{I-1} M_i(x_i, p)$. Since $p \cdot x' < \sum_i M_i(x_i, p)$, it follows that $p \cdot x'_i < M_i(x_i, p)$, so $x'_i \not\succ x_i$ and $x'$ is not a Pareto improvement over $x$.

We repeat that the conclusion of Theorem 2(a) holds even under cases 2 and 3 of Remark 4: a competitive equilibrium allocation maximizes the money-metric sum even if the money metric does not represent preferences of some consumers. The sum can
still be used to identify candidates for optimal allocations.\footnote{Bergstrom (1973), Gale and Mas-Colell (1977), Fon and Otani (1979), Weymark (1985b), and Rigotti and Shannon (2005) present versions of the first welfare theorem with incomplete or intransitive preferences.}

**Remark 5** (Money-metrics and Bernheim-Rangel’s (2009) unambiguous choice relations). Bernheim and Rangel (2009) propose two unambiguous choice relations that extend the usual revealed-preference relation. Their purpose is to facilitate welfare analysis for non-standard (behavioral) decision theories in which choices violate the weak axiom of revealed preference—and so any rationalizing binary relation is either incomplete or intransitive. They work in a setting of a choice function defined over a family of constraint sets which includes all finite constraint sets of an underlying “consumption set.” They define an unambiguously chosen over relation \( P^* \) by \( x P^* y \) if \( y \) is never chosen if \( x \) is available. They also define a weakly unambiguously chosen over relation \( R' \) by \( x R' y \) if, whenever \( x \) is available and \( y \) is chosen, \( x \) is also chosen.

The money metric in case 3. in the preceding Remark 4 can be used to uncover whether alternatives do or do not stand in these relations. Specifically, (i) if \( p \cdot y < M(x, p) \) and \( \succeq \) is complete, then \( x \succ y \) and therefore \( x P^* y \); and (ii) if \( p \cdot y < M(x, p) \) and \( \succeq \) is not necessarily complete, then either \( x \succ y \) and therefore \( x P^* y \), or \( x \) and \( y \) are not comparable, in which case it cannot be that \( y R' x \). See Figure 3.

\[ \square \]

**Remark 6** (A Representative Consumer?). A natural question arises as to the sense which the sum of money-metrics gives us a “representative consumer.” Remark ?? suggests a limitation of this interpretation. A positive representative consumer refers to a hypothetical consumer who owns all the economy’s wealth and has a preference relation that generates the economy’s aggregate demand at every price-aggregate wealth pair. Maximizing the sum of money-metrics at a price-aggregate wealth pair generates a set consisting of every vector of demands generated by any distribution of that aggregate wealth (Corollary 2). The italicized words indicate the differences. An example might help further illustrate the differences. Consider an exchange economy with two consumers and two goods. Consumer \( i \) only cares for good \( i \), the more the better. Consumer \( i \)’s endowment is a fraction \( \alpha_i \) of the aggregate endowment \( \omega_i = \alpha_i \omega \). (That is, endowments are collinear.) This economy has an aggregate demand generated by Cobb-Douglas preferences: \( U(x_1, x_2) = x_1^{\alpha_1} x_2^{\alpha_2} \), so clearly there is a positive representative consumer for this economy. The representative consumer has homothetic preferences. The economy, however, is clearly is not Gorman: the consumers do not have wealth expansion paths that are straight lines with common slope. The sum of money-metric utilities is \( M = p_1 x_{11} + p_2 x_{22} \) where \( x_{ii} \) is consumer \( i \)'s consumption of good \( i \). Maximizing this sum subject to \( p_1 x_1 + p_2 x_2 \leq w = p_1 \omega_1 + p_2 \omega_2 \) gives all aggregate consumption plans that meet the budget constraint (with consumer \( i \) getting all of good \( i \)).
“money-metric aggregate consumer” is indifferent between Pareto Optimal allocations at a given aggregate wealth level. That “consumer’s” maximum value is simply aggregate wealth at any such allocation.  

Remark 7 (Sum of money metrics and price changes). One objection to choosing allocations to maximize the sum of non-concave money-metrics is that small changes in reference price vectors might result in large changes in allocations (Blackorby and Donaldson 1988, p. 129). But in a smooth economy with locally unique equilibria—for example the exchange economy in Debreu (1970) with $C^1$ demand functions satisfying a boundary condition—we have this: as prices vary smoothly with the endowment, equilibrium allocations vary smoothly with them since demands are $C^1$. When prices vary endogenously with the endowments, there is no discontinuity in allocations that maximize the money-metric sum.

5 Inequality and the Bosmans-Decancq-Ooghe Rehabilitation

We now turn to the sense in which the non-quasiconcavity of the money-metric sum leads to inegalitarian allocations. We begin with an example that illuminates this sense and use it to explain the relationship between our rehabilitation of money-metrics and the one offered in Bosmans, Decancq, and Ooghe (2018), henceforth BDO.

Example 1 (Money-metrics and inequality). Consider an economy consisting of two consumers with identical preferences $\succsim$ on $X = \mathbb{R}^L_+$ that satisfy the Standard Assumptions, and in addition are strongly convex and strictly monotone ($x > y$ implies that $x \succ y$), but not quasi-homothetic; and let $M(\cdot, p)$ be their common continuous, non-concave (at some $p$) money-metric. Then we know that it is not mid-point concave at...
some \( p \), which is to say,

\[
2M\left(\frac{1}{2}x + \frac{1}{2}y, p\right) < M(x, p) + M(y, p)
\]

for some \( x \) and \( y \in X \). \( \square \)

Let the aggregate endowment be \( x + y \). Then the sum of money metrics is higher at the unequal allocation in which one consumer gets \( x \) and the other gets \( y \) than if they each get \( (x + y)/2 \). Since the consumers have the same convex preferences, equal division is Pareto optimal, and so one of the consumption bundles, say \( x \), is worse than an equal division of the aggregate resources.

Our rehabilitation does not rule out this example. But we note that it depends on the reference price. By Theorem 2, if we consider a reference price \( p^\ast \) that supports equal division as a competitive equilibrium allocation in this exchange economy, then the sum of those money-metrics would not increase with a move to an unequal allocation: The inegalitarian implication here can be undone by suitable choice of the reference price.

The BDO rehabilitation consists of a representation theorem for a social preference relation, that is, a relation which specifies, for each possible profile \( R \) of preferences in an economy, a binary relation \( \succsim_R \) over allocations. It asserts that social preferences satisfy six axioms if and only if there is a reference price vector \( p \) and a Schur-concave function\(^{38} \) \( W \) such that, for each preference profile \( R \),

\[
x' \succsim_R x \quad \text{if and only if} \quad W(M_1(x'_1, p), \ldots, M_I(x'_I, p)) \geq W(M_1(x_1, p), \ldots, M_I(x_I, p)).
\]

The money-metric sum is intended to capture efficiency, the Schur-concave function \( W \), equity.

Since the sum of money metrics is a special case of the representation they prove,\(^{39} \) the BDO rehabilitation likewise does not rule out Example 1. Their novel axiom, the one that they use to capture equity concerns, is referred to as the Efficiency-Preserving Transfer Principle. To explain it, consider an economy in which two of the consumers in it have the same preferences, and an allocation in which one of these consumer is richer than the other in the sense that one consumer has more of every good than another. The efficiency-preserving transfer principle asserts that if the richer consumer transfers some consumption to the poorer consumer (but remains weakly richer), and the Scitovsky set for the economy remains unchanged by the transfer, then the new allocation is weakly preferred in the social order.\(^{40} \)

\(^{38}\)A function \( f \) on a nonempty convex set \( D \subset \mathbb{R}^n \) is Schur-concave if \( f(x) \geq f(y) \) whenever \( y \) majorizes \( x \), that is, whenever \( \sum_k y_{[k]} \geq \sum_k x_{[k]} \) for \( k = 1, \ldots, n - 1 \), with equality for \( k = n \), where \( y_{[n]} \geq y_{[n-1]} \geq \cdots \geq y_{[1]} \) and \( x_{[n]} \geq x_{[n-1]} \geq \cdots \geq x_{[1]} \).

\(^{39}\)The function \( f(y) = \sum y_i \) is (weakly) Schur-concave.

\(^{40}\)The Scitovsky set for a population of consumers \((1, \ldots, I)\) at a reference allocation \((x_1, \ldots, x_I)\) is the sum of at-least-as-good-as sets from this allocation. Formally, at an allocation \((x_1, \ldots, x_I)\) it equals \( \{x' \in X | x' = \sum_{i=1}^I x'_i\} \).
How does Example 1 escape this axiom? One of two ways. Either the preferences of the consumers are quasi-homothetic, in which case their money-metrics are concave and the example cannot arise; or they are not quasi-homothetic. In the second case, the Scitovsky sets generally vary as the allocation of a fixed stock of goods changes, and so the axiom does not apply.\footnote{Perhaps a more descriptively-apt name for their axiom is the Scitovsky-set-preserving transfer principle.} Indeed, Gorman (1953) proved that the Scitovsky set is globally invariant to changes in the allocation of a stock of goods if and only if preferences of consumers are quasi-homothetic with a common slope of their wealth-expansion paths.\footnote{To be sure, Bosmans, Decancq, and Ooghe (2018) write clearly in their Footnote 14 that “This axiom does not cover all cases where the distributions before and after the transfer are both efficient (equal marginal rates of substitution). Indeed, in some such cases the Scitovsky boundaries do not coincide, but rather intersect at the societal bundle.”} If the Scitovsky set changes with a reallocation, as it in general does in Example 1, then the BDO axioms are silent about the desirability of an inequality-increasing transfer (even when \(x >> y\) and we perturb slightly the equal allocation as their axiom requires).

As a final point of difference, the BDO rehabilitation is one in the social choice register, as opposed to welfare economics. This is to say, BDO consider axioms on how social rankings over allocations behave over difference profiles of consumer preferences, rather than for a fixed society.\footnote{In particular, to have bite, their efficiency-preserving transfer principal requires that the economy have at least two consumers with the same preferences.}

6 Money-metric Relations: Benefit and Distance functions

We now consider the relationship between money metric utility and two other functions: the benefit function of Luenberger (1992a) and the distance function analyzed by Gorman (1970).\footnote{The distance function was introduced in producer theory by Shepard (1953) and in consumer theory by Malmquist (1953); Deaton (1979) reviews some of the history of this function.} Throughout this subsection, we impose the Standard Assumptions on preferences for each consumer and assume that each consumer \(i\)’s preferences has a utility representation, \(u_i\) on \(X_i\). We point out that versions of Theorems 1 and 2 hold for these functions. We refer the reader to Luenberger (1992a) and Gorman (1970) for proofs of the properties of the benefit and distance functions that we assert here.\footnote{Gorman (1970) is reprinted in Blackorby and Shorrocks (1995); proofs of the properties we use here can also be found in Deaton (1979) and Luenberger (1995).} We refer the reader to Luenberger (1992a) and Gorman (1970) for proofs of the properties we assert here.

To elaborate, let \(x_i \in X_i, \bar{u}_i \in \text{Range}(u_i), g \in \cap_{i=1}^I X_i\) (assuming the intersection is nonempty). Define \(b_i(x_i, \bar{u}_i),\) consumer \(i\)’s benefit function at \((x_i, \bar{u})\), to be the number \(\beta\) which solves \(u(x_i - \beta g) = \bar{u}_i\) if a solution exists, \(-\infty\) otherwise. And define \(f_i(x_i, \bar{u}_i),\)
consumer $i$’s distance function at $(x_i, \bar{u}_i)$ to be the number $\gamma$ which solves $u_i(x_i/\gamma) = \bar{u}_i$ if a solution exists, $-\infty$ otherwise. To convert these numbers into money units, define $B_i(x_i, \bar{u}_i, p) = b_i(x_i, \bar{u}_i, p) \cdot g$ and $F_i(x_i, \bar{u}_i, p) = f_i(x_i, \bar{u}_i)e_i(p, \bar{u}_i)$.\footnote{Gorman (1970) works in a setting for which the distance function is real-valued.}

Parallel to Theorem 1, one can show that if $x_i^* \in d_i(p, w_i)$, then $(x_i^*, 1)$ solves the saddlepoint inequalities with either $B_i(x_i, V_i(p, w), p)$ or $F_i(x_i, V_i(p, w), p)$ replacing the consumer’s money-metric in the Lagrangian. Luenberger (1992a) proved the first, without using the language of saddlepoints. As a corollary it follows that $x_i^*$ maximizes both $B_i(x_i, V_i(p, w), p) - p \cdot x_i$ and $F_i(x_i, V_i(p, w), p) - p \cdot x_i$ on $X_i$—again ignoring the budget constraint. An important lemma in the proof of this fact spells out the relationship between the expenditure functions and the benefit and distance functions:

$$e_i(p, \bar{u}_i) = \min_{x_i \in X_i} \{p \cdot x_i/f_i(x_i, \bar{u}_i)\} = \min_{x_i \in X_i} \{p \cdot x_i - b_i(x_i, \bar{u}_i)\},$$

where, for the first equality, prices are normalized so that $p \cdot g = 1$.

We know of no proof of the fact that if $x_i^* \in d_i(p, w_i)$ then $x_i^*$ maximizes $F_i(x_i, V_i(p, w), p) - p \cdot x_i$ on $X_i$. We sketch it here. Clearly $p \cdot x_i^* = w$ and $f(x_i^*, V(p, w_i)) = 1$, so the objective equals 0 at $x_i^*$. Substitute $V_i(p, w_i)$ into the utility argument on both sides of the first equality in (9) to get $w = \min_{x_i \in X_i} \{p \cdot x_i/f(x_i, V_i(p, w_i))\} \leq p \cdot x_i^*/f(x_i^*, V_i(p, w_i))$. For any affordable $x_i^*$, $p \cdot x_i^* \leq w$, so the preceding inequality implies that $f(x_i^*, V_i(p, w_i))w_i \leq p \cdot x_i^*$, so the objective is at most 0 for any affordable $x_i^* \in X_i$.  Define $B(x, \bar{u}) = \sum_i B_i(x_i, \bar{u}_i)$ and $F(x, \bar{u}) = \sum_i F_i(x_i, \bar{u}_i)$, where $\bar{u} = (\bar{u}_1, ..., \bar{u}_I)$ is a utility profile. Parallel to Theorem 2, if $(x^*, y^*, p^*)$ is a competitive equilibrium, then $(x^*, y^*)$ maximizes $B(x, u^*) + p^* \cdot y - p^* \cdot x$ and $F(x, u^*) + p^* \cdot y - p^* \cdot x$ on $\mathcal{A}$, where $u_i^* = V_i(p^*, w_i(p^*))$ for $i = 1, ..., I$. Luenberger (1992b) proves this last fact for the aggregate benefit function.\footnote{Luenger (1992a) normalizes prices so that $p \cdot g = 1$, so the extra multiplicative term drops out.}

We can now present

**Proposition 1.** The conclusion of Theorem 2(a) continues to hold if each $M_i$ is replaced by $F_i$; or each $M_i$ is replaced by $B_i$.

For special cases $F_i(x_i, V_i(p, w), p)$ and $B_i(x_i, V_i(p, w), p)$ equal the money-metric $M_i(x_i, p)$ (sometimes up to an additive constant). For $X_i = \mathbb{R}_+^\ell$, $F_i(x_i, V_i(p, w), p) = M_i(x_i, p)$ when $i$’s preferences are homothetic (the consumer’s wealth drops out of $F_i$ when preferences are homothetic); and $B_i(x_i, V_i(p, w), p) = M_i(x_i, p) + w$ when $i$’s preferences are quasi-linear with respect to some good $\ell$ and the reference bundle $g$ equals the $\ell$-th unit vector. Unlike the money-metric function, the benefit and distance functions are concave whenever preferences are convex, and so they avoid the inegalitarian

\footnote{It is hard to believe that the aggregate distance function $F(x, \bar{u}^*)$ has not be used by someone, and that the Proposition has not been proved for it, but we do not know a reference for it.}
implications of Example 1. Like the money-metric function, the benefit and distance functions need not represent preferences: as already mentioned, there are convex preference relations not representable by a concave function (Kannai (1977)). They each also fail property (5) that the money metric satisfies. It is easy to show that the aggregate benefit and distance functions are not Pareto consistent: a change can increase the measure, but every person is worse off with the change. Such examples are easy to construct in single-consumer economies even for preferences that are Cobb-Douglas (for the benefit function) or quasilinear (for the distance function).

7 Money-metrics and Cost-Benefit Analysis in the Small

We now consider how the money-metric sum behaves for small changes in an aggregate production plan. In particular we show that the derivative of the sum of money metrics equals a local welfare measure proposed by Radner (1993 [1978]). Since Radner showed that a positive sign of this measure implies that a small change in the aggregate production is a potential Pareto improvement, this result gives a foundation for using first-order approximations to money metric utility (for example, Deaton (2003)). According, in the next section we add the assumption that, for each consumer $i$, $M_i(\cdot, p)$ is differentiable in $x$ at a particular point $x_i^0$ in the interior of $X_i$ satisfying $x_i^0 \in d_i(p, p \cdot x_i^0)$. In subsection 9.2, we give sufficient conditions on $\succ_i$ for such differentiability to hold. Although we assume for that result that $\succ_i$ is complete and strongly convex, we do not assume that it is transitive.

7.1 Radner’s Local Welfare Measure

We now remind the reader of Radner’s (1993 [1978]) local welfare measure. He considers an economy in which each consumer’s preferences are represented by a $C^2$, concave, nonsatiated function (implicitly on a convex consumption set with nonempty interior). The last two conditions of course imply local nonsatiation. Let $\tilde{y} : [0, 1] \to Y$, where $Y \subseteq \mathbb{R}^L$ is a connected, nonempty production set. Radner (1993) refers to the scalar $\alpha$ as indexing projects. Suppose that $\tilde{y}$ is differentiable at $\alpha = 0$. The total supply for project $\alpha$ is $\omega + \tilde{y}(\alpha)$. A feasible consumption allocation of the total supply $\omega + \tilde{y}(\alpha)$ is a point $(x_1, ..., x_I) \in X_i$ such that $\sum_i x_i \leq \omega + \tilde{y}(\alpha)$. A feasible consumption allocation $x = (x_1, ..., x_I)$ is a valuation equilibrium relative to a price $p \in \mathbb{R}_{++}^L$ if, for each $i = 1, ..., I$, $x_i \in d_i(p, p \cdot x_i)$. We stress that the model is silent about how the aggregate production plan is chosen; in particular it need not be part of a competitive equilibrium. The valuation equilibrium assumption assures that the consumption allocation of a given
supply is Pareto optimal: the source of any inefficiency in the economy is the aggregate production plan.

Radner (1993 [1978]) proposed this local measure of welfare,

\[ p^0 \cdot \tilde{y}'(0), \quad (10) \]

where the feasible allocation \( \tilde{x}(0) = (\tilde{x}_1(0), \ldots, \tilde{x}_I(0)) \) of the initial supply \( \omega + \tilde{y}(0) \) is a valuation equilibrium with respect to \( p^0 \). He assumes that each \( \tilde{x}_i(0) \) is in the interior of \( i \)'s consumption set, so the consumer’s first-order conditions hold as an equality.

Equation (10) is the derivative of \( p^0 \cdot (\tilde{y}(\alpha) - \tilde{y}(0)) \) at \( \alpha = 0 \); in words, it is the change in the value of the aggregate production plan measured by the initial valuation equilibrium prices. He goes on to give conditions under which, if (10) is positive, then for some \( \alpha > 0 \), the supply \( \omega + y(\alpha) \) is a potential Pareto improvement over the supply \( \omega + y(0) \) for every \( \alpha \in (0, \bar{\alpha}) \): the total supply of \( \omega + y(\alpha) \) can be allocated so that each consumer \( i \)'s prefers its consumption in this allocation to \( \tilde{x}_i(0) \).

We next relate the derivative of the sum of money-metrics to Radner’s measure (10).

### 7.2 The Money-metric Sum and Radner’s Measure

In what follows let \( \tilde{x}(\alpha) = \omega + \tilde{y}(\alpha) \) be a consumption allocation of the supply \( \omega + \tilde{y}(\alpha) \). Define \( M(\alpha) = \sum_i M_i(\tilde{x}_i(\alpha), p^0) \), the sum of money-metric utilities. The next result asserts that \( M'(0) \) equals Radner’s local welfare measure. Here \( \text{int}(X_i) \) refers to the interior of \( X_i \).

**Proposition 2** (The money-metric sum and Radner’s measure). Suppose that (a) each \( \succeq_i \) satisfies either the Standard or the Alternative assumptions; (b) \( \tilde{x}(0) \in \text{int}(\Pi_iX_i) \) and \( \tilde{x}(\cdot) \) is differentiable at \( \alpha = 0 \); (c) each \( M_i(\cdot, p^0) \) is differentiable at \( x_i = \tilde{x}_i(0) \); and (d) the allocation \( \tilde{x}(0) \) is a valuation equilibrium relative to some \( p^0 \in \mathbb{R}_L^+ \). Then

\[ M'(0) = p^0 \cdot \tilde{y}'(0). \]

We use the next easy fact in the proof.

**Lemma 2.** Suppose that \( \succeq_i \) satisfies either the Standard or Alternative assumptions. Fix \( x^0 \in \text{int}(X_i) \) and \( p^0 \in \mathbb{R}_L^+ \) with \( x^0 \in d_i(p, p \cdot x^0) \). If \( M_i(\cdot, p^0) \) is differentiable at \( x = x^0 \), then

\[ D_x M_i(x^0, p^0) = p^0. \quad (11) \]

**Proof of Lemma 2.** Since \( x^0 \in d_i(p^0, p \cdot x^0) \), \( x^0 \) maximizes \( M_i(x, p^0) - p^0 \cdot x \) on \( X_i \) by Theorem 1(a). And since \( M_i(\cdot, p^0) \) is differentiable at \( x^0_i \), the necessary Kuhn-Tucker

\[ 50 \text{If the measure is negative, the supply at all small-enough } \alpha \text{ is not a potential Pareto improvement over allocation at } \alpha = 0. \text{ This observation is a special case of case 3 in Remark 4.} \]
conditions must hold at \( x_i = x^0_i \). Since \( x^0 \in \text{int}(X_i) \), the Kuhn-Tucker conditions hold as equalities, and (11) holds. \( \square \).

**Proof of Proposition 2**: By Lemma 2, and (a)-(d), each \( M_i(\tilde{x}_i(\alpha), p^0) \) is differentiable at \( \alpha = 0 \) and the derivative equals \( p^0 \cdot \tilde{x}'_i(0) \). Sum over all consumers and use feasibility to find \( M'(0) = \sum_i p^0 \cdot \tilde{x}'_i(0) = p^0 \cdot \tilde{y}'(0) \). \( \square \)

Under the additional assumptions that for each \( \alpha \in [0, 1] \), \((\tilde{x}(\alpha), \tilde{p}(\alpha))\) is a competitive equilibrium of an exchange economy in which consumer \( i \)’s endowment is \( \tilde{x}_i(0) + (1/I)(\tilde{y}(\alpha) - \tilde{y}(0)) \); and \( \tilde{p}(\cdot) \) differentiable at \( \alpha = 0 \), Radner (1993 [1978], p. 136) proved that if \( p^0 \cdot \tilde{y}'(0) > 0 \), then for all small-enough \( \alpha > 0 \), each consumer \( i \) strictly prefers \( \tilde{x}_i(\alpha) \) to \( \tilde{x}_i(0) \): the new supply \( \omega + \tilde{y}(\alpha) \) is a strict potential Pareto improvement over the initial allocation; that is, the new supply can be allocated so that each consumer \( i \) prefers its new consumption to \( \tilde{x}_i(0) \). That conclusion follows here under the Standard Assumptions by equation (5). With incomplete or intransitive preferences, (5) can fail. The difficulty, we should stress, is not restricted to money metrics. Rather, it is how to pass from the inequality \( p^0 \cdot \tilde{y}'(0) > 0 \) to the implication that a small-enough increase in \( \alpha \) is a potential strict Pareto improvement.

**Proposition 3** (Local Potential Pareto Improvements). In addition to the hypotheses of Proposition 2, assume that (e) for each consumer \( i \), \( \succsim_i \) is complete; (f) for every \( \alpha \in [0, 1] \), \((\tilde{x}(\alpha), \tilde{p}(\alpha))\) is a competitive equilibrium for the exchange economy in which consumer \( i \)’s endowment is \( \omega_i(\alpha) = \tilde{x}_i(0) + (1/I)(\tilde{y}(\alpha) - \tilde{y}(0)) \) and \( \tilde{p}(0) = p^0 \); and (g) \( \tilde{p}(\cdot) \) is differentiable at \( \alpha = 0 \).

If \( M'(0) = p^0 \cdot \tilde{y}'(0) > 0 \) then for every \( \alpha' \) in some nonempty interval \((0, \bar{\alpha})\), the total supply at \( \alpha' \) is a strict potential Pareto improvement over the allocation at \( \alpha = 0 \).

Before giving the proof we point out that the differentiability assumptions on the competitive equilibrium made in Proposition 3 can be justified from primitives.

**Remark 8.** In the economy with endowments indexed by \( \alpha \in [0, 1] \) as in Proposition 3, the already-mentioned Remark of Debreu (1970, p. 390) implies that, if each consumer has a \( C^1 \) demand (satisfying a boundary condition), then unless the point \( \tilde{x}(0) \) happens to lie in a closed subset of \( \mathbb{R}_+^n \) of Lebesgue-measure zero, an exchange economy with endowment \( \tilde{x}(0) \) has a finite number of equilibria; and as the endowment varies with \( \alpha \), each of these equilibrium price vectors is continuously differentiable on an interval including \( \alpha = 0 \), which justifies (g). If we pick one of these and set \( p^0 \) equal to the equilibrium price selected at \( \omega + y(0) \), it follows that each consumer’s equilibrium consumption for this selection is continuously differentiable on this interval, justifying conditions (b) and (f). Since Debreu (1970) works directly with demand, rather than preferences, the conclusion is unaffected by preference intransitivity. In particular, Al-Najjar (1993) identifies
a class of smooth intransitive preferences that generate $C^1$ demands and so can satisfy the demand conditions in Debreu (1970).

**Proof of Proposition 3:** Consider a consumer $i$ whose preferences satisfy the Standard Assumptions. Certainly $\bar{x}_i(\alpha) \succ_i^* \omega_i(\alpha)$ for every $\alpha \in [0, 1]$. Let $M_i(\alpha) = M_i(\omega_i(\alpha), p^0)$. By Lemma 2 have $M_i'(0) = p^0 \cdot \bar{y}'(0)/I > 0$, which implies $M_i(\alpha) > M_i(0)$ for consumer $i$ for every $\alpha$ in some interval $(0, \alpha_i)$. We should add that consumer inertia and indeterminacy: their consumers all have incomplete preferences of intransitive nature, but does not touch on local changes, the subject of Proposition 3.

Now consider a consumer $i$ who does not satisfy the Standard Assumptions, and so satisfies the Alternative Assumptions with $\succ_i$ complete. It follows from Shafer (1974, Theorem 2) that consumer $i$’s demand $d_i$ is nonempty, single-valued and gives the best affordable point in the budget set: in particular if $x = d_i(p, w)$ and $p \cdot y \leq w$ with $y \neq x$, then $x \succ_i y$.

Differentiate both sides with respect to $\alpha$, evaluate at $\alpha = 0$ and use $\bar{p}(0) \cdot \bar{y}'(0) > 0$ to find that $\bar{p}(0) \cdot \bar{x}'_i(0) > 0$. It follows that $\bar{p}(\alpha) \cdot \bar{x}_i(\alpha) > \bar{p}(0) \cdot \bar{x}_i(0)$ for all $\alpha \in (0, \alpha_i)$ for some $\alpha_i > 0$. Since $\bar{x}_i(\alpha) = d_i(\bar{p}(\alpha), \bar{w}_i(\alpha))$, it follows that $\bar{x}_i(\alpha) \succ_i \bar{x}_i(0)$ for every $\alpha \in (0, \alpha_i)$. Take $\bar{\alpha} = \min\{\alpha_1, ..., \alpha_I\}$. Then $\bar{x}_i(\alpha) \succ_i \bar{x}_i(0)$ for $i = 1, ..., I$ and $\alpha \in (0, \bar{\alpha})$.

Proposition 3 extends Radner’s (1993 [1978]) theorem to a setting with intransitive preferences. We do not extend it to incomplete preferences for two reasons. First, the differentiability assumptions on the equilibrium allocations, prices, and the money metrics are likely to fail for an economy populated by consumers with incomplete preferences: typically, with incomplete preferences, for each price there is more than one demand; and for each consumption plan there is more than one (normalized) price that supports it as a demand. More seriously, our proof does not extend to consumers with incomplete preferences: the “revealed preference” argument that any demand point is strictly preferred to any other affordable point, does not hold for consumers with incomplete preferences. And it is easy to find examples of economies with incomplete preferences for which the Radner measure is positive, but the conclusion fails. For example, it can fail for the economy that Rigotti and Shannon (2005) use to illustrate consumer inertia and indeterminacy: their consumers all have incomplete preferences of the class proposed by Bewley (2002) that has “kinks everywhere.” We should add that both the assumptions and the conclusion of Proposition 3 can easily hold for economies with consumers who have intransitive preferences; see Remark 8.

\footnote{Fountain (1981) points out difficulties for standard welfare measures for changes “in the large” when consumers have intransitive preferences, but does not touch on local changes, the subject of Proposition 3.}
7.3 Other Welfare Measures in the Small

Schlee (2013a) shows that Radner’s local measure equals the derivatives of four other welfare measures, including aggregate consumers’ surplus. The three other measures are the Slutsky change in real income, the nominal Divisia price index, and a nominal version of Debreu’s (1951) coefficient of resource utilization. As Schlee (2013a) points out, the derivative of the five measures also equal the derivatives of the compensating and equivalent variations, bringing the total of locally equivalent measures to seven. Schlee (2018) points out that the local equivalence between Radner’s measure and the coefficient of resource utilization is false.52

Proposition 2 and Corollary 3 extend these equivalences to the sum of money-metric utilities.

One can also show that the local equivalence extends to the aggregate benefit and distance functions of Section 6. This follows since the conclusion of Lemma 2 holds for these functions: \( \nabla_x B_i(x_i, V_i(p, w), p) = p = \nabla_x F_i(x_i, V_i(p, w), p) \) whenever an interior point \( x_i \) is demanded at \( (p, w) \). The last paragraph of Schlee (2018) gives the details.

As already mentioned, both Blackorby and Donaldson (1987) and Hammond (1994) develop new welfare measures as alternatives to the sum of money metrics, at least partly in response to the non-concavity of individual money metrics. Blackorby and Donaldson (1987) propose an alternative that they call welfare ratios. For consumer \( i \) with utility representation \( u_i \) it is given by \( r_i(p, w) = \frac{w_i}{e(p, \bar{u}_i)} \), where \( \bar{u}_i \) is taken to be a poverty level of utility for consumer \( i \). It can be converted into money units by multiplying \( r_i \) by \( e(p^0, \bar{u}_i) \) for some reference price \( p^0 \) (following Deaton (2003, equation (8))). Letting \( R_i(p, w, p^0) \) denote the resulting product, it follows that \( \nabla_x R_i(p, p \cdot x_i, p^0) = p^0 \) when the derivative is evaluated at \( (p^0, p^0 \cdot x_i, p^0) \) and the local equivalence extends to the sum of such nominal welfare ratios. Hammond’s (1994) aggregate money-metric measure is the number \( \mu \) given by

\[
W(V_1(p, w_1), ..., V_I(p, w_I)) = W(V_1(p^0, w_1^0 + \mu), ..., V_I(p^0, w_I^0 + \mu))
\]

where \( V_i \) is consumer \( i \)'s indirect utility function, \( W \) is some strictly increasing function of the utility profile, and the price-wealth pairs to the right of the equality describe some baseline allocation. This measure is generally not locally equivalent to the derivative of the sum of money-metrics.53 This failure is not an oversight of the author’s: The function \( W \) is intended to incorporate equity concerns and so, unlike the money-metric sum, rank some non-equilibrium allocations higher than competitive equilibrium allocations.

---

52The correct equivalence is between Radner’s measure and a measure related to the coefficient that takes as a reference point a production plan that need not be a part of a Pareto optimal allocation, rather than the entire aggregate production set.

53This follows from his equation (39) after suppressing the public good and imposing budget balance on the right side. Only in the special in which his \( \omega_i \) weights are equal does the local equivalence go through.
8 Concluding Remarks on Future Work

We conclude with two remarks on future directions that are opened up by our rehabilitation of the money-metric. The first concerns the notion of complementarity; and the second, comparative-static analysis of the consumer’s problem. Both will draw on the version of the Kuhn-Tucker-Uzawa saddlepoint theorem on non-linear programming presented here.

Samuelson (1974) used the money-metric to offer two new definitions of complementary commodities (his fifth and sixth definitions), and thereby connect to basic and classical themes in consumer theory. This program we carry forward in future work. Thus our next step is to reconsider his definitions of complementarity in light of our Corollary 1, and relate it to other proposed definitions in the literature. Of particular interest are those suggested or explored by Chipman (1977) and Kannai (1980), and more recently by Chambers, Echenique, and Shmaya (2010).

Of perhaps greater interest is the use of the money-metric in reconsidering results in monotone comparative statics of Quah (2007) and Mirman and Ruble (2008). A important tool to carry this out is our Corollary 1, which formulates the consumer’s problem as one subject only to non-negativity constraints, bypassing the budget constraint altogether. This point is of obvious importance since one stumbling block in applying lattice methods to the consumer’s problem is that the budget set is not a lattice; more to the point, different budget sets are not comparable in the strong-set order.

9 Appendix

9.1 Omitted proofs

Proof of Lemma 1: Fix $x \in X$ satisfying the local cheaper point condition at $p >> 0$. Since $p >> 0$, $\succsim$ is closed, and $X$ is closed and bounded from below, both $h(p,x)$ and $d(p,M(x,p))$ are nonempty. Let $x' \in h(p,x)$ and $x'' \in d(p,M(x,p))$. Since $p \cdot x' = M(x,p)$ and $\succsim$ is complete, $x'' \succsim x'$. By $x' \succsim x$ and transitivity, $x'' \succsim x$. Since by local nonsatiation, $p \cdot x'' = M(x,p)$, $x'' \in h(p,x)$. And since $x$ satisfies the local cheaper point assumption, there is a sequence $x^n$ in $X$ with limit $x''$ and $p \cdot x^n < p \cdot x'' = M(x,p)$, so, by completeness, $x \succ x^n$ for every $n$. By closedness of $\succsim$, $x \succ \lim x^n = x''$, so $x'' \succsim x' \succsim x \succsim x''$ and both $x' \in d(p,p \cdot x)$ and $x' \sim x$ follow from transitivity. □

Proof of Theorem 1(b): If the Saddlepoint inequalities (6) hold, then by Theorem 1 in Uzawa (1958), $x^*$ maximizes $M(\cdot, p)$ on $B(p,w)$. If $M(\cdot, p)$ represents preferences on $X$—for example if $X = \mathbb{R}^k_+$—then we would be done by Uzawa’s (1958) Theorem 1. Since we consider more general consumption sets, we proceed using Lemma 1.
Let \( x' \in h(p, x^*) \), so by Lemma 1, \( x' \sim x^* \) and \( x' \in d(p, M(p, x^*)) \). The inequalities \( \mathcal{L}(x^*, 1) \leq \mathcal{L}(x^*, \lambda) \) for every \( \lambda \geq 0 \) imply that \( p \cdot x^* = w \), so \( x^* \in B(p, w) \). And the inequality \( \mathcal{L}(x', 1) \leq \mathcal{L}(x^*, 1) \) implies that \( M(x', p) - p \cdot x' \leq M(x^*, p) - p \cdot x^* \). Since \( x' \sim x^* \), \( M(x', p) = M(x^*, p) \), so \( \mathcal{L}(x, 1) \leq \mathcal{L}(x^*, 1) \) reduces to \( w = p \cdot x^* \leq p \cdot x' \). Since \( x^* \sim x' \succsim x \) for every \( x \in B(p, w) \) and \( \succsim \) is transitive, \( x^* \in d(p, w) \). \( \square \)

**Proof of Theorem 2(b):** Now suppose that the saddlepoint inequalities (7) hold at some \( p^* \gg 0 \). That \( \mathcal{L}_{p^*}(x^*, y^*, p^*) \leq \mathcal{L}_{p^*}(x^*, y^*, \mu) \) for every \( \mu \in \mathbb{R}_{++}^p \) and \( p^* \gg 0 \) implies that \( \omega + y^* - x^* = 0 \). That \( \mathcal{L}_{p^*}(x, y, p^*) \leq \mathcal{L}_{p^*}(x^*, y^*, p^*) \) for every \((x, y) \in \mathcal{A}\) immediately implies that \( p^* \cdot y_j^* \geq p^* \cdot y_j \) for every \( y_j \in Y_j \) for every firm \( j \). Since \( p^* \gg 0 \), \( \omega > 0 \), and \( 0 \in Y_j \), \( p^* (\omega + y^*) > 0 \). Let \( v_i \) be given by \( p^* \cdot x_i^* = v_i(p^* \cdot \omega + p^* \cdot y^*) \) and set \( w_i^* = v_i(p^* \cdot \omega + p^* \cdot y^*) \). Set consumer \( i \)'s endowment of goods equal \( v_i \omega \) and \( i \)'s ownership share of each firm equal to \( v_i \), so \( i \)'s wealth is \( w_i^* \). That \( x_i^* \in d_i(p^*, w_i^*) \) now follows from Theorem 1 (b). \( \square \)

**9.2 On the Differentiability of a Money-metric**

We give preference conditions for \( M_i(\cdot, p) \) to be differentiable at a point \( x \in d(p, p \cdot x) \). We use such differentiability in Section 7. Let \( g_i(x) = \{ p \in \mathbb{R}_{++}^L \mid \sum p_i = 1 \text{ and } x \in d_i(p, p \cdot x) \} \), the normalized inverse demand correspondence for consumer \( i \).

**Proposition 4** (Money metric differentiability). Suppose that \( X_i \subseteq \mathbb{R}_+^L \) is closed and convex with nonempty interior, and that \( \succsim_i \) is complete and satisfies the Alternative Assumptions. Let \( p^0 \in g_i(x^0) \) with \( x^0 \in \text{int}(X_i) \). Then \( M_i(\cdot, p^0) \) is differentiable at \( x = x^0 \) if in addition

(a) \( g_i(x) \) is nonempty and single-valued for every \( x \) in some open neighborhood \( N_a \subset X \) of \( x^0 \);

(b) \( g_i(x) \) is Lipschitz continuous at the point \( x^0 \): namely, for some open neighborhood \( N \subset N_a \) of \( x^0 \), there is a real number \( K > 0 \) such that, for every \( x \in N \)

\[
\|g_i(x^0) - g_i(x)\| \leq K\|x^0 - x\|
\]

Before turning to a proof, we make a few remarks. First, we do not assume that preferences are transitive. Second, we can of course replace (b) with the assumption that \( g_i(\cdot) \) is differentiable in a neighborhood of \( x^0 \). Third, **strong** convexity of preferences is not necessary. If preferences on \( X = \mathbb{R}_+^L \) are represented by

\[
u(x) = \sum_{\ell} x_{\ell},
\]

31
then \( M(x,p) = \min \{p_1,...,p_L\} \sum_{\ell} x_{\ell}, \) which is differentiable in \( x. \) Fourth, if the inverse demand \( g_i(x^0) \) is not unique, then the conclusion can fail.\(^{54}\) It remains to be seen how far Lipschitz continuity of the inverse demand at a demand point can be relaxed. In a follow-up paper on comparative statics with the money metric, we confirm that if \( \succsim \) has an increasing \( C^1 \) utility representation with no critical point on the consumption set \( X = \mathbb{R}^L_{++}, \) then the money metric is monotone and \( C^1 \) with no critical point. In this case \( g_i \) is clearly continuous but not necessarily Lipschitz-continuous-at-a-point.

We use the next fact in the proof of Proposition 4. The conclusion is standard, but we do not assume transitivity.\(^{55}\)

**Lemma 3.** If \( \succsim_i \) is complete and satisfies the Alternative Assumptions, then the compensated demand \( h_i(p,x) \) defined in (1) is continuous at every \((p,x) \in \mathbb{R}^L_{++} \times X_i.\)

**Proof of Lemma 3.** Suppress the consumer subscript throughout. We will show that the lemma is a consequence of Berge’s theorem.\(^{56}\) Since \( X \) is closed, \( p \gg 0 \) and \( \succsim \) is closed and strongly convex, \( h(p,x) \) is nonempty and single-valued for every \((p,x) \in \mathbb{R}^L_{++} \times X.\) Fix \((p^0,x^0) \in \mathbb{R}^L_{++} \times X.\) Let \( K = \{x \in X \mid x_\ell \leq 1 + p^0 \cdot x_\ell/p^0_\ell, \ell = 1,...,L \}. \) Since \( X \) is closed, \( K \) is compact and \( x^0 \in K. \) Moreover for some neighborhood \( N \) of \( x^0 \) and \( P \) of \( p^0, \) \( h(p,x) \in K. \) This follows since for any \( y = h(p,x), y_\ell \leq M(x,p)/p_\ell \leq p \cdot x/\ell. \) For \( x \in N, \) let \( R(x) = \{y \in K \mid y \succsim x\}. \) Then for \((p,x) \in P \times N, h(p,x) = \arg\min_{x^0 \in R(x)} (p \cdot x). \) We now show that the correspondence \( R(\cdot) : N \Rightarrow K \) is continuous at \( x^0. \) Since \( \succsim \) is closed, \( R \) is upperhemicontinuous. To prove that it is lowerhemicontinuous, let \( y^0 \in R(x^0) \) and consider any sequence \( x^n \) in \( X \) converging to \( x^0. \) We need to construct a sequence \( y^n \) converging to \( y^0 \) with \( y^n \in R(x^n) \) for every \( n. \) If \( x^0 = y^0, \) then \( y^n = x^n \) for all \( n \) suffices. If \( x^0 \neq y^0, \) then for every integer \( n \) set \( y^n = x^n \) if \( x_n \succsim \lambda y^n_0 + (1-\lambda)x_0^0 \) for every \( \lambda \in [0,1]; \) otherwise set \( y^n = \lambda_n y^n_0 + (1-\lambda_n)x^0, \) where \( \lambda_n = \sup \{\lambda \in [0,1] \mid \lambda y^n_0 + (1-\lambda)x^0 \succsim x^n\}. \) Since \( \succsim \) is closed, \( y^n \in R(x^n) \) for every \( n. \) Moreover, since \( \succsim \) is open and \( \succsim \) is strongly convex, there is an integer \( n^* \) such that \( y^n = \lambda_n y^n_0 + (1-\lambda_n)x^0 \) for every \( n > n^*. \) If \( y^n \rightarrow y^0, \) then there is a subsequence of \( \lambda_n \) that is bounded away from one, with a further subsequence \( \lambda_{k(n)} \) with limit \( \bar{\lambda} < 1. \) Since \( \bar{\lambda} = \bar{\lambda} y^n_0 + (1-\bar{\lambda})x^0 \succsim x^0, y^k(n) \rightarrow \bar{y}, \) \( x^{k(n)} \rightarrow x^0 \) and \( \succsim \) is open, \( y^{k(n)} \notin R(x^{k(n)}) \) for all large enough \( n, \) a contradiction. So \( y^n \rightarrow y^0 \) and \( R \) is continuous at \( x^0. \) By Berge’s theorem, \( h \) is continuous at \((p^0,x^0). \)

**Proof of Proposition 4:** We omit the consumer \( i \) subscript throughout. By Lemma 2, if \( M(\cdot,p) \) is differentiable at \( x' \in d(p,p \cdot x') \) with \( x' \in \text{int}(X), \) then \( D_{x}M(x',p) = p. \) We

---

\(^{54}\)The case of homothetic preferences in which the homogenous-of-degree-1 representation is not differentiable makes this point clear: the money metric takes the form \( u(x)b(p) \) where \( u \) is homogenous of degree 1. If \( u \) is not differentiable at a demand point, then neither is the money-metric (e.g. Leontief preferences).

\(^{55}\)Honkapohja (1987) proves that the compensated demand is upper hemicontinuous in \((p,x)\) for transitive, but merely convex preferences. Transitivity is used in an essential way in his proof.

\(^{56}\)For a statement of Berge’s theorem, see for example Kreps (2013, p. 476).
will show that
\[
\lim_{\|x-x^0\|\to 0} \frac{|M(x^0, p^0) - M(x, p^0) - p^0 \cdot (x^0 - x)|}{\|x - x^0\|} = 0. \tag{12}
\]

From the Saddlepoint inequalities of Theorem 1 we have
\[
M(x^0, p^0) - M(x, p^0) \geq p^0 \cdot (x^0 - x), \tag{13}
\]
for every \(x \in X\), so for any \(x \in X\) with \(x \neq x^0\)
\[
M(x^0, p^0) - M(x, p^0) - p^0 \cdot (x^0 - x) \geq 0. \tag{14}
\]
For \(x \in N\), we have \(M(x, g(x)) = g(x) \cdot x\) and—recalling that \(g(x)\) is a singleton on \(N—M(x^0, g(x)) < g(x) \cdot x^0\) whenever \(x \neq x^0\). From these inequalities find that
\[
M(x^0, g(x)) - M(x, g(x)) < g(x) \cdot (x^0 - x) \tag{15}
\]
which, adding and subtracting the same terms, is the same as
\[
M(x^0, p^0) - M(x, p^0) - p^0 \cdot (x^0 - x) < (g(x) - p^0) \cdot (x^0 - x) - \Gamma(x) \tag{16}
\]
where \(\Gamma(x) = [M(x^0, g(x)) - M(x^0, p^0)] + [M(x, p^0) - M(x, g(x))]\). Since the compensated demand \(h(p^0, x)\) is single-valued, \(M(x, \cdot)\) is differentiable with\(^{57}\)
\[
D_pM(x, p^0) = h(p^0, x).
\]

By the mean value theorem, for each \(x\) there is a number \(\alpha(x, y) \in [0, 1]\) such that, for \(y = x\) or \(y = x^0\)
\[
M(y, p^0) - M(y, g(x)) = h(\alpha(x, y)g(x) + (1 - \alpha(x, y))p^0, y) \cdot (g(x) - p^0) \tag{17}
\]
Let \(q(x, y) = \alpha(x, y)g(x) + (1 - \alpha(x, y))p^0\) for \(y = x, x^0\). In this notation
\[
[M(x^0, g(x)) - M(x^0, p^0)] + [M(x, p^0) - M(x, g(x))] = [h(q(x, x^0), x^0) - h(q(x, x), x)] \cdot (g(x) - p^0) \tag{18}
\]
Insert (18) into (16) to find
\[
M(x^0, p^0) - M(x, p^0) - p^0 \cdot (x^0 - x) < (g(x) - p^0) \cdot (x^0 - x + h(q(x, x^0), x^0) - h(q(x, x), x)) \tag{19}
\]

\(^{57}\)This standard “envelope” conclusion follows by changing notation and the names in standard proofs of the differentiability of the cost function (the McKenzie-Shepard lemma), in spite of the lack of transitivity. See Fuchs-Seliger (1990b, Theorem 2).
Combine (14) and (19) to find

\[ |M(x^0, p^0) - M(x, p^0) - p^0 \cdot (x^0 - x)| \leq \Omega(x) \quad (20) \]

where \( \Omega(x) = (g(x) - p^0) \cdot (x^0 - x + h(q(x, x^0), x^0) - h(q(x, x), x)) \). Since \( g(x^0) = p^0 \), and \( h \) is continuous by Lemma 3, it follows from the Lipschitz condition (b) and the Cauchy-Schwartz inequality that

\[
\lim_{\|x - x^0\| \to 0} \frac{\Omega(x)}{\|x - x^0\|} = 0,
\]

so (12) follows.

### 9.3 A Representation Theorem

We slightly extend the representation theorem in Khan and Schlee (2016). This theorem differs from those in Weymark (1985a) in that we do not impose convexity of the consumption set. Fix \( p >> 0 \). corollary. Let \( X^d(p) \subseteq X \) be the set of consumption plans satisfying \( x \in d(p, p \cdot x) \), that is, plans \( x \) that are demanded at \( (p, p \cdot x) \).

**Theorem 3.** Let \( X \subseteq \mathbb{R}_+^L \) be nonempty and closed. If \( \succ \in X \times X \) satisfies the Standard Assumptions, and \( p >> 0 \), then \( M(\cdot, p) \) represents \( \succ \) on \( X^{\text{lcp}}(p) \cup X^d(p) \).

**Lemma 4.** If \( x \in X^d(p) \), and \( p >> 0 \), then the conclusion of Lemma 1 holds for \((x, p)\); that is, (a) \( d(p, M(x, p)) = h(p, x) \), and (b) if \( y \in d(p, M(x, p)) \) then \( y \sim x \).

**Proof of Lemma 4:** Suppose that \( y \in h(p, x) \), so that \( p \cdot y = M(x, p) \) and \( y \succ x \); it follows that \( y \in d(p, M(x, p)) \). Suppose that \( y \in d(p, M(x, p)) \), so \( y \succ x \). By local nonsatiation, \( p \cdot y = M(x, p) \), so \( y \in h(p, x) \). Since \( x \in d(p, M(x, p)) \), it follows that \( y \sim x \). \( \square \)

We can now complete the proof of Theorem 3.

**Proof of Theorem 3.** Suppose that \( x \succ y \). By transitivity, \( \{z \in X \mid z \succ y\} \subseteq \{z \in X \mid z \succ y\} \), so \( M(x, p) \geq M(y, p) \). Now suppose that \( y \succ x \). By Lemmata 1 and 4, \( z' \in d(p, M(z, p)) \) implies that \( z' \sim z \) for \( z = x, y \). So \( y' \sim y \succ x \sim x' \). By transitivity, \( y' \succ x' \). By LNS, \( M(y, p) > M(x, p) \). \( \square \)
References


