Campaign Spending, Media Capture, and Political Accountability

Vinícius Tiné
Rafael Costa Lima

March 1, 2019

Abstract

This paper is concerned with the role of electoral campaign spending in political accountability. Using a political agency model, campaign spending is established as a fundamental part of a mechanism which links the objectives of lobbies to political accountability. The analysis makes very clear that campaign spending allows lobbies to influence both the selection of politicians and their behavior while in office, which in turn are intimately related to political accountability. Furthermore, it is shown that the possibility of media capture by lobbies is also a part of that mechanism, as it affects the attractiveness of campaign spending. Consequently, decisions on campaign spending and media capture by lobbies are interdependent.

Keywords: Lobbying, Political Campaign, Rent-Seeking.
JEL: D72
1 Introduction

Economists have extensively studied and discussed a range of possibilities of citizen welfare improvement through government intervention. The provision of public goods, and the regulation of externalities and market power are traditional examples of such opportunities. However, there are no guarantees that public decision-making would follow those footsteps. Driving it closer to the public interest is a central issue in modern democracies.

As pointed out by Besley (2006), selecting public-spirited officials and designing an institutional framework that affects the incentives facing the public decision-maker are two fundamental pieces in achieving political accountability. In turn, the relative success on these tasks depends on recognizing different forces which may interact with public decision-makers, as well as the channels through which these interactions take place.

One potentially important influence over policy decisions comes from the acting of lobbies, and it is typically channeled by campaign financing. By spending resources on electoral campaigns, lobbies may exert a strong pressure on public decisions, in the event in which their chosen candidates are successful in the given election.

This paper is concerned with the role of campaign spending in political accountability. Using a political agency model pioneered by Barro (1973) and Ferejohn (1986), it establishes that campaign spending has an outstanding role as part of a mechanism which links the objectives of lobbies to political accountability. Campaign spending allows lobbies to influence both the selection of politicians and their behavior while in office, which in turn are intimately related to political accountability. Offering this insight in a very clear way through the model is the main contribution of the paper to the literature on political accountability.
The basic political agency model describes democratic politics as the strategic interaction between voters and politicians. Voters decide whether to reelect an incumbent politician or to select a newcomer for the job, in a setting where asymmetric information about politicians’ types is an usual feature. In that context, the present paper incorporates a key idea into the model, which is a mutually advantageous deal between an office-seeking politician and a rent-seeking lobby. On one side, a lobby spends resources on the office-seeking politician’s electoral campaign. That campaign spending is assumed to affect that politician’s probability of victory. On the other hand, if elected, that politician will be fully committed to follow the lobby’s instructions on policy decisions. The introduction of those new elements into the model builds a path between the objectives of lobbies and political accountability, and campaign spending is a fundamental piece of that construction.

The paper also contributes to the understanding of the role of media capture in political accountability. According to Besley and Prat (2006), media capture affects voters’ information, and hence their voting decisions, which in turn affect public decisions. This perspective can also be obtained through our model, in which, besides campaign spending, media capture is treated endogenously. Nevertheless, the present paper highlights a new indirect relationship between the extent of media capture and political accountability through campaign spending. As it is demonstrated, the possibility of media capture by lobbies compose the attractiveness of campaign spending, and hence indirectly affects political accountability.

Since mainly the 1970’s, many studies have provided valuable insights about the theme of political accountability, using the tools of modern economics. In a very important paper for that literature, Barro (1973) utilized a canonical political agency model to underline political income as a possible source of conflict between politicians and voters, leading to policy choices that are not in the best public interest. In that context, it was shown that periodic elections can
be an effective mechanism to improve the equilibrium of the model in favour of voters. More than a decade later, Ferejohn (1986) offered an important complementary study to the former, extending the analysis to an infinite horizon case. Those two papers have set the basic ideas and the methodological foundations for the further development of the literature. Besley and Case (1995) have shown that when voters take into account both the performances of their incumbent politician and of the incumbent politician in a neighboring jurisdiction, an interesting yardstick competition emerges between those, and that competition may affect political accountability. Coate and Morris (1995) examine the role of asymmetric information about policies and politicians in the choice of inefficient methods of resource redistribution to special interests. Persson, Roland, and Tabellini (1997) have analysed the role of separation of powers in political accountability. It was demonstrated that, under appropriated checks and balances conditions, separation of powers improves the accountability of elected officials.

The remaining sections of the paper are organized in the following way. Section 2 introduces the model and presents the analysis of its equilibrium. First, it is presented a simpler version of the model, in which the strategic interaction between lobbies and politicians through campaign spending is highlighted. Then, we allow for the possibility of media capture, offering a richer description of that interaction. The analysis provides a very clear picture of the joint role of campaign spending and media capture in both selection and discipline of politicians, which compose political accountability. Section 3 offers results on comparative statics and welfare, using variations on key parameters of the model. Those results reinforce the description of the interplay between campaign spending and media capture, along with their connection to political accountability. Finally, Section 4 brings concluding remarks.
2 The Model

2.1 The Baseline Setup

Our basic political agency model is very close in its structure to the form presented in Besley (2006). There are two time periods, viewed as political terms and indexed by $t \in \{1, 2\}$. The first political term starts with an electoral dispute between two politicians of different types: a congruent (or public-spirited) politician versus a dissonant (or egoistic) politician. The winner of the election is typically drawn randomly by nature in such a model and the probability of victory of the congruent politician is exogenously given as $\pi \in [0, 1]$. Once declared the victor, the incumbent politician for the first term has to make a single policy decision, denoted by $e_1 \in \{0, 1\}$. After this decision is made, the voters decide whether to reelect the incumbent politician straight to the second term or to allow a second election to take place at the beginning of the second period, between two fresh politicians of once again different types. Once the incumbent politician for the second term is determined, a second and last policy decision $e_2 \in \{0, 1\}$ is made. Voters do not observe the incumbent politician’s type.

For each $t$, there is a state of nature $s_t \in \{0, 1\}$, which is observed only by the incumbent politician, with equal probabilities for each possible value. Each agent’s payoff depends on the policy decision, which is made with the state of nature in mind. Voters receive $\Delta$ if $e_t = s_t$ or zero if $e_t = 1 - s_t$, and congruent politicians share voters' preferences. Then, if the incumbent politician is congruent, he always chooses $e_t = s_t$. Dissonant politicians, on the other hand, have different preferences. If $e_t = s_t$, they receive zero. However, if $e_t = 1 - s_t$, they receive a dissonant rent, denoted by $r_t$, which is the realization of a random variable. $r_t$ is continuously distributed on the interval $[0, R]$, has an expected value of $\mu$, a pdf $g(\cdot)$, and a cdf $G(\cdot)$. The realization of $r_t$ is revealed to the incumbent just before he has to decide on $e_t$. Moreover, $r_1$ and $r_2$ are
independent. Finally, all agents discount the future with $\beta \in (0, 1)$.

This model is excellently suited for the present work, mainly for two particular reasons. First, its flexibility permits incorporating campaign spending in its structure. Second, its simplicity allows for a very clear analysis of the role of campaign spending in attaining political accountability.

We now introduce the first distinctive ingredient of the present work, as we allow the probability of victory of the dissonant politician in each election to be determined endogenously in the model, instead of the usual exogenous probability assumption.

2.2 Lobbying and Campaign Spending

We assume that the probability of victory in the election $t$ of the dissonant politician, denoted by $1 - \pi_t$, is a function of his campaign effort, denoted by $b_t$, given by

$$1 - \pi_t = \frac{b_t}{b_t + 1} \tag{1}$$

A lobby is now introduced as a new agent in the model. The lobby emerges as a part of a mutually advantageous deal with the dissonant politician: it provides funds for the dissonant politician’s electoral campaign. If elected, the dissonant politician will follow the lobby’s instructions about policy decisions. The amount of campaign effort $b_t$ is chosen by the lobby, and each choice of $b_t$ has an associated cost of $c \cdot b_t$ for the lobby, where $c > 0$. The amount $c \cdot b_t$ should be interpreted as the lobby’s campaign spending on election $t$. It is assumed that $c < \mu$, that is, the constant marginal cost $c$ is not too high, in order to obtain interior solutions to the lobby’s decision problems. Finally, the lobby shares the dissonant politician’s preferences perfectly and also observes $r_t$ at the moment when the incumbent must decide on $e_t$.

On the other hand, the congruent politician does not make any deal
with lobbies. His probability of winning the election $t$ will be a mere consequence of the lobby’s decision of $b_t$, i.e.,

$$
\pi_t = \frac{1}{b_t + 1}
$$

(2)

It is useful to outline the timing of the model:

- The lobby decides on the campaign effort $b_1$
- Nature reveals the dissonant rent $r_1$
- The lobby decides on the policy instruction $e_1$
- Voters’ decision
- The lobby decides on the campaign effort $b_2$
- Nature reveals the dissonant rent $r_2$
- The lobby decides on the policy instruction $e_2$

### 2.3 Equilibrium with Campaign Spending

We solve the model for a unique Bayesian equilibrium. If the dissonant politician is the incumbent on the second term, the lobby always instructs $e_2 = 1 - s_2$. Therefore, the analysis starts at the following decision.

**The lobby decides on the campaign effort $b_2$**

Suppose that the lobby must decide on the campaign effort for an electoral campaign on $t = 2$. His problem is

$$
\max_{b_2 \geq 0} \frac{b_2}{b_2 + 1} \cdot \mu - cb_2
$$

(3)

The solution for that problem is

$$
b_2^* = \sqrt{\frac{\mu}{c} - 1}
$$

(4)
It will be useful to define
\[ \alpha \equiv \frac{b_2^*}{b_2^* + 1} \mu - cb_2^* = \mu - 2\sqrt{\mu c} + c \] (5)

**Voters' decision**

Voters decision works similarly as in Besley (2006). Let \( \lambda \) be the probability that the dissonant incumbent chooses \( e_1 = s_1 \) and let \( \Pi \) be the probability that the incumbent is congruent given that \( e_1 = s_1 \) was chosen. If the incumbent on the first term chooses \( e_1 = 1 - s_1 \), voters know that it can only be a dissonant politician, as congruent politicians always choose \( e_1 = s_1 \). Then, they do not re-elect the incumbent and a second election takes place at \( t = 2 \). On the other hand, if \( e_1 = s_1 \) is chosen, voters re-elect the incumbent if
\[ \Pi \equiv \frac{\pi_1}{\pi_1 + (1 - \pi_1) \lambda} \geq \pi_2^* \] (6)

We rearrange the inequality above and state it as follows:

**Condition 1.**
\[ \frac{1}{1 + b_1 \lambda} \geq \frac{1}{b_2^* + 1} \] (7)

In fact, the above condition holds (see Appendix 1 for a proof). Hence, the first period incumbent is re-elected if \( e_1 = s_1 \) is chosen.

**The lobby decides on the policy instruction \( e_1 \)**

Suppose that the dissonant politician is the incumbent on \( t = 1 \). If the lobby instructs \( e_1 = s_1 \), the incumbent will be re-elected and the lobby will instruct \( e_2 = 1 - s_2 \) on the second term. Then, his payoff is equal to \( \beta \mu \). However, if he instructs \( e_1 = 1 - s_1 \), the incumbent will not be re-elected and a new election takes place, when \( b_2^* \) will be chosen for the new dissonant politician’s campaign. In this case, lobby’s payoff is equal to \( r_1 + \beta \alpha \). Thus, it is clear that the lobby instructs \( e_1 = s_1 \) if
\[ r_1 + \beta \alpha < \beta \mu, \] (8)
i.e., if
\[ r_1 < \beta (\mu - \alpha) \] (9)

This occurs with a probability of \( \lambda = G (\beta (\mu - \alpha)) \). Hence, the lobby instructs \( e_1 = 1 - s_1 \) if \( r_1 > \beta (\mu - \alpha) \), which occurs with a probability of \( 1 - \lambda = 1 - G (\beta (\mu - \alpha)) \).

The lobby decides on the campaign effort \( b_1 \)

The lobby’s problem of deciding on the campaign effort \( b_1 \) is therefore

\[
\max_{b_1 \geq 0} \frac{b_1}{b_1 + 1} \cdot \left\{ \lambda \beta \mu + (1 - \lambda) \left[ E (r_1 \mid r_1 > \beta (\mu - \alpha) \right] + \beta \alpha \right\} \\
- cb_1 + \frac{1}{b_1 + 1} \beta \alpha
\] (10)

In order to simplify the notation, let \( \tilde{\rho} \equiv \beta (\mu - \alpha) \). Then, we can rewrite the problem more concisely as

\[
\max_{b_1 \geq 0} \frac{b_1}{b_1 + 1} \cdot \tilde{k} - cb_1 + \beta \alpha,
\] (11)

where \( \tilde{k} \equiv \lambda \tilde{\rho} + (1 - \lambda) E (r_1 \mid r_1 > \tilde{\rho}) \). The solution is

\[
\tilde{b}_1 = \sqrt{\tilde{k} - \frac{c}{\tilde{k}}} - 1
\] (12)

The equilibrium analysis above establishes in a very clear way the role of campaign spending as a link between the objectives of the lobby and policy decisions. As was shown, through campaign spending, the lobby has a direct influence on both political selection and discipline, respectively represented by \( \pi_t \) and \( \lambda \). It also consists in a simple description of the strategic interaction between lobbies and politicians.
2.4 Media Capture

As in Besley (2006), we allow for a probability $\tau$ that the type of the dissonant incumbent is revealed to voters at the moment of their re-election decision. However, we introduce the assumption that $\tau$ is a function of the amount of the lobby’s influence over the media, denoted by $m$:

$$\tau = \frac{1}{m + 1} \quad (13)$$

The amount $m$ is decided by the lobby just before voters’ decision. Of course, the lobby would like $\tau$ to be the lowest possible, but the choice of $m$ has an associated cost of $\eta \cdot m$, where $\eta > 0$. We assume that $\beta(\mu - \alpha) > \eta$, that is, the constant marginal cost $\eta$ is not too high. The probability $\tau$ has the following interpretation. A low value of $\tau$ means that the lobby has a strong control of the media.

Once again, the timing of the model is outlined:

- The lobby decides on the campaign effort $b_1$
- Nature reveals the dissonant rent $r_1$
- The lobby decides on the policy instruction $e_1$
- The lobby decides on the influence over the media $m$
- Voters’ decision
- The lobby decides on the campaign effort $b_2$
- Nature reveals the dissonant rent $r_2$
- The lobby decides on the policy instruction $e_2$

We now proceed to the equilibrium analysis of the full model with campaign spending and media capture.
2.5 Equilibrium with Campaign Spending and Media Capture

Here, lobby’s decision on $b_2$ remains the same as before, i.e., $b_2^* = \sqrt{\mu} - 1$. Therefore we go straight to voters’ decision on re-election.

Voters’ decision

Now, at the moment of voters’ decision, four different scenarios are possible if the incumbent is dissonant:

(1) $e_1 = 1 - s_1$ and the incumbent’s type is revealed.

(2) $e_1 = 1 - s_1$ and the incumbent’s type is not revealed.

(3) $e_1 = s_1$ and the incumbent’s type is revealed.

(4) $e_1 = s_1$ and the incumbent’s type is not revealed.

However, it is only in scenario (4) that there is the possibility of the dissonant incumbent being re-elected. In fact, voters do re-elect the incumbent in that scenario, which is the same as in section 2.3. In what follows, it is that particular scenario which is kept in mind.

The lobby decides on the influence over the media $m$

The lobby knows that the dissonant incumbent is re-elected if and only if $e_1 = s_1$ is chosen and the incumbent’s type is not revealed. Hence, if $e_1 = 1 - s_1$ is chosen, lobby’s payoff is equal to $r_1 + \beta \alpha$, as before. On the contrary, if $e_1 = s_1$ is chosen, lobby’s payoff will depend on his choice of $m$ and will be equal to

$$\max_{m \geq 0} \{(1 - \tau) \beta \mu + \tau \beta \alpha - \eta m\}$$

(14)
This expression is maximized with

\[ m^* = \frac{\sqrt{\beta (\mu - \alpha)}}{\sqrt{\eta}} - 1 \]  

(15)

The lobby decides on the policy instruction \( e_1 \)

Suppose that the dissonant politician is the incumbent on \( t = 1 \). If the lobby instructs \( e_1 = s_1 \), the incumbent will be re-elected with a probability of \( 1 - \tau^* = \frac{m^*}{m^* + 1} \). Then, the lobby’s expected payoff would be equal to \((1 - \tau^*) \beta \mu + \tau^* \beta \alpha - \eta m^* \). However, if he instructs \( e_1 = 1 - s_1 \), the incumbent will not be re-elected and a new election takes place, when \( b^*_2 \) will be chosen for the new dissonant politician’s campaign. In this case, lobby’s payoff is equal to \( r_1 + \beta \alpha \). Thus, it is clear that the lobby instructs \( e_1 = s_1 \) if

\[ r_1 + \beta \alpha < (1 - \tau^*) \beta \mu + \tau^* \beta \alpha - \eta m^* , \]  

(16)

i.e., if

\[ r_1 < \beta (1 - \tau^*) (\mu - \alpha) - \eta m^* \]  

(17)

This occurs with a probability equal to

\[ \lambda = G (\beta (1 - \tau^*) (\mu - \alpha) - \eta m^*) \]  

(18)

The lobby decides on the campaign effort \( b_1 \)

The lobby’s problem of deciding on the campaign effort \( b_1 \) is therefore

\[
\max_{b_1 \geq 0} \frac{b_1}{b_1 + 1} \cdot \left\{ \lambda [(1 - \tau^*) \beta \mu + \tau^* \beta \alpha - \eta m^*] + (1 - \lambda) [E (r_1 | r_1 > \beta (1 - \tau^*) (\mu - \alpha) - \eta m^*) + \beta \alpha] \right\}
- cb_1 + \frac{1}{b_1 + 1} \beta \alpha 
\]

(19)

As before, in order to simplify the notation, let \( \rho \equiv \beta (1 - \tau^*) (\mu - \alpha) - \eta m^* \). Then, we can rewrite the problem much more concisely as

\[
\max_{b_1 \geq 0} \frac{b_1}{b_1 + 1} \cdot k - cb_1 + \beta \alpha ,
\]

(20)
where $k \equiv \lambda \rho + (1 - \lambda) E(r_1 \mid r_1 > \rho)$. As will be shown later (see Appendix 2, Part (iv), Step 2.2), $k > \mu$. This yields the solution

$$b_1^* = \sqrt{\frac{k}{c}} - 1 \quad (21)$$

The preceding model with campaign spending and media capture has several strong points. First, it is a rich yet simple description of the interaction between lobbies and politicians. Second, it captures in a clear way the joint role of campaign spending and media capture in political accountability. Finally, it specifically unveils an indirect connection between media capture and political accountability, through campaign spending, thus contributing to the discussion on media capture represented by Besley and Prat (2006).

3 Comparative Statics and Welfare Analysis

In this section, we analyse the effects of an increase on each parameter $c$ and $\eta$ on the model’s variables and on voter welfare. Such exercise offers an alternative angle on the interdependence of campaign spending and media capture in the model, as well as on the role of these variables in political accountability.

3.1 Comparative Statics

In this first part, we present two propositions.

**Proposition 1.** (The effects of an increase on the constant marginal cost of the electoral effort $c$)

(i) An increase on $c$ causes a decrease in the probability of victory of the dissonant politician in an election on the second term.

(ii) An increase on $c$ causes a decrease in the probability $\tau^*$ that the dissonant incumbent’s type is revealed to voters.
(iii) An increase on $c$ causes an increase on discipline $\lambda$.

(iv) An increase on $c$ causes a decrease in the probability of victory of the dissonant politician in an election on the first term.

Proof: See Appendix 2.

**Proposition 2.** (The effects of an increase on the constant marginal cost of the influence over the media $\eta$)

(i) An increase on $\eta$ does not affect the electoral effort $b_2$. Consequently, the probability of victory of the dissonant politician in an election on the second term remains unaltered.

(ii) An increase on $\eta$ causes an increase in the probability $\tau^*$ that the dissonant incumbent’s type is revealed to voters.

(iii) An increase on $\eta$ causes a decrease in discipline $\lambda$.

(iv) An increase on $\eta$ causes a decrease in the probability of victory of the dissonant politician in an election on the first term.

Proof: See Appendix 3.

Among the preceding results, part (ii) in Proposition 1 and part (iv) in Proposition 2 are especially interesting, as both concern the interdependence between campaign spending and media capture. As $c$ increases, it becomes even less attractive to the lobby to have his incumbent kicked out of office after the first term, once running a second-term election becomes less profitable. Then, an increase in media capture is a result of stronger incentives to remain in power. On the other hand, an increase in $\eta$ means a diminished attractiveness of media capture, thus also diminished incentives for campaign spending on the first election.
3.2 Welfare Analysis

As in Besley (2006), we define the welfare function for the electorate. Voters’ ex-ante welfare on the first term is given by

\[ V_1(\lambda, \tau^*) = [\pi_1^* + (1 - \pi_1^*) \lambda] \Delta \] (22)

Similarly, voters’ ex-ante welfare on the second term is given by

\[ V_2(\lambda, \tau^*) = \pi_1^* \Delta + (1 - \pi_1^*) (1 - \lambda) \pi_2^* \Delta + (1 - \pi_1^*) \lambda \tau^* \pi_2^* \Delta, \] (23)

that is,

\[ V_2(\lambda, \tau^*) = [\pi_1^* + (1 - \pi_1^*) \pi_2^* [(1 - \lambda) + \lambda \tau^*]] \Delta \] (24)

Finally, we define voters’ welfare function as

\[ W(\lambda, \tau^*) = V_1(\lambda, \tau^*) + \beta V_2(\lambda, \tau^*), \] (25)

or, alternatively, as

\[ W(\lambda, \tau^*) = [\pi_1^* + (1 - \pi_1^*) \lambda] \Delta + \beta [\pi_1^* + (1 - \pi_1^*) \pi_2^* [(1 - \lambda) + \lambda \tau^*]] \Delta \] (26)

Equation (27) is a very concise way to show how campaign spending and media capture are related to political accountability, through political selection and discipline.

The next result concerns the effects of an increase on the marginal cost \( c \) of campaign effort on voters’ welfare. It consists of the net effect of the results presented on Proposition 1.

Proposition 3. (Welfare Analysis)

An increase on \( c \) causes an increase on voters’ welfare.
Proof: See Appendix 4.

On the other hand, the net effect of an increase on the marginal cost \( \eta \) of the influence over the media on voter’s welfare has an unclear sign. The procedure which was utilized to aggregate the effects of an increase on \( c \) and support Proposition 3 proves to be not effective to similarly aggregate the effects of an increase in \( \eta \).

4 Conclusion

The main message provided by this paper is that campaign spending opens a channel through which lobbies can influence public decision-making. Furthermore, it is demonstrated that the possibility of media capture also contributes to that influence.

Although campaign spending decisions are part of a mutually advantageous deal between lobbies and politicians, such decisions are shown to have important externalities over the political process. Therefore, the paper reinforces the importance of thinking rigorously about campaign spending from a regulatory perspective. As documented by Scarrow (2007), the introduction of campaign spending limits has been a common regulatory practice in the last decades, mainly in developed countries. In a recent study, Avis et al. (working paper) provides empirical evidence on the effects of campaign spending limits in Brazil, which were introduced incorporated in the year of 2015. They find that higher spending limits decreases political competition and increases incumbency advantage in elections.

As a contribution to that literature on the regulatory perspective on campaign spending, we intend to incorporate the following additional exercise to the present paper. First, binding caps on both the campaign spending variable \( b \) and the influence over the media \( m \) will be introduced in the model. Then,
it will be possible to determine the effect of increasing those constraints over political accountability. This will be an important exercise, from the regulatory point of view.

As an additional robustness exercise, we intend to analyse the equilibrium of the model without assuming that the dissonant incumbent politician will be fully committed to the lobby’s instructions. Through such an exercise, it will be possible to determine whether the message of the paper depends on that simplifying assumption or not.
5 Appendix

5.1 Appendix 1

Proof. Suppose that $\Pi < \pi^*_2$, i.e., that
\[
\frac{1}{1 + b_1 \lambda} < \frac{1}{b_2^* + 1}
\] (27)

Then, voters will not reelect an incumbent which chooses the congruent policy in $t = 1$. Thus, as there will not be incentives for discipline, lobby’s problem of choosing $b_1$ will be
\[
\max_{b_1 \geq 0} \frac{b_1}{b_1 + 1} \cdot (\mu + \beta \alpha) - cb_1 + \frac{1}{b_1 + 1} \cdot \beta \alpha
\] (28)

The solution is
\[
b_1^* = \frac{\sqrt{\mu}}{\sqrt{c}} - 1 = b_2^*
\] (29)

By assumption, we have
\[
\frac{1}{1 + b_2^* \lambda} < \frac{1}{b_2^* + 1},
\] (30)
a contradiction. $\Box$

5.2 Appendix 2

(Part (i))

(Step 1: $\frac{\partial b_2^*}{\partial c} < 0$)

Proof. We know that $b_2^* = \sqrt{\frac{c}{c}} - 1$. Then,
\[
\frac{\partial b_2^*}{\partial c} = -\frac{\sqrt{c}}{2c^{3/2}} < 0
\] (31)

$\Box$
(Step 2: $\frac{\partial(1-\pi_2^*)}{\partial c} < 0$)

Proof. As $\pi_2^* = \frac{1}{b_2^*+1} = \frac{\sqrt{\mu}}{\sqrt{\eta}}$, we have

$$\frac{\partial \pi_2^*}{\partial c} = \frac{1}{2\sqrt{\mu} \sqrt{c}} > 0 \quad (32)$$

Consequently,

$$\frac{\partial(1 - \pi_2^*)}{\partial c} = -\frac{\partial \pi_2^*}{\partial c} < 0 \quad (33)$$

\(\Box\)

(Part (ii))

(Step 1: $\frac{\partial \alpha}{\partial c} < 0$)

Proof. $\alpha$ can be written as

$$\alpha = \mu - 2\sqrt{\mu c} + c \quad (34)$$

Then, we have

$$\frac{\partial \alpha}{\partial c} = -\frac{\sqrt{\mu}}{\sqrt{c}} + 1 = -\left[\frac{\sqrt{\mu}}{\sqrt{c}} - 1\right] = -b_2^* < 0 \quad (35)$$

\(\Box\)

(Step 2: $\frac{\partial m^*}{\partial c} > 0$)

Proof. As $m^* = \frac{\sqrt{\eta(\mu-\alpha)}}{\sqrt{\eta}} - 1$, we have

$$\frac{\partial m^*}{\partial c} = \frac{\sqrt{\beta}}{\sqrt{\eta}} \cdot \frac{1}{2\sqrt{\mu - \alpha}} \cdot \left(\frac{\partial \alpha}{\partial c}\right) = \frac{\sqrt{\beta}}{\sqrt{\eta}} \cdot \frac{b_2^*}{2\sqrt{\mu - \alpha}} > 0 \quad (36)$$

\(\Box\)

(Step 3: $\frac{\partial \tau^*}{\partial c} < 0$)
Proof. As \( \tau^* = \frac{1}{1+m^*} \), we have

\[
\frac{\partial \tau^*}{\partial c} = \frac{\partial \tau^*}{\partial m^*} \cdot \frac{\partial m^*}{\partial c} = - \frac{1}{(1+m^*) \cdot \beta (\mu - \alpha)} \frac{\sqrt{\beta}}{\sqrt{\eta}} \frac{b^*}{2(\mu - \alpha)} = \frac{\sqrt{\eta}}{\sqrt{\beta (\mu - \alpha)}} \cdot \frac{b^*}{2(\mu - \alpha)} = \frac{\sqrt{\eta}}{\beta (\mu - \alpha)} \cdot \frac{b^*}{2(\mu - \alpha)} < 0
\]  

\( \text{(Part (iii))} \)

Proof. We know that \( \lambda = G(\rho) \), where \( \rho = \beta (1 - \tau^*) (\mu - \alpha) - \eta m^* \). Then, in order to determine \( \frac{\partial \lambda}{\partial c} \), we will proceed by studying \( \frac{\partial \rho}{\partial c} \). After some rearranging, \( \rho \) can be written as

\[
\rho = \left( \sqrt{\beta (\mu - \alpha)} - \sqrt{\eta} \right)^2
\]  

Then, we have

\[
\frac{\partial \rho}{\partial c} = 2 \left( \sqrt{\beta (\mu - \alpha)} - \sqrt{\eta} \right) \cdot \frac{\partial}{\partial c} \left( \sqrt{\beta (\mu - \alpha)} \right) = 2 \left( \sqrt{\beta (\mu - \alpha)} - \sqrt{\eta} \right) \cdot \frac{1}{2\sqrt{\beta (\mu - \alpha)}} : \beta \cdot \left( \frac{\partial \alpha}{\partial c} \right)
\]  

\( \geq 0 \)

Thus,

\[
\frac{\partial \lambda}{\partial c} > 0
\]  

\( \text{(Part (iv))} \)

Proof. This demonstration involves several steps.

\( \text{(Step 1: } \frac{\partial k}{\partial c} > 0 \text{)} \)
Proof. $k$ was defined as $k \equiv \lambda \rho + (1 - \lambda) E(r_1 \mid r_1 > \rho)$. Therefore,

$$\frac{\partial k}{\partial c} = \left[ \frac{\partial \lambda}{\partial c} \cdot \rho + \lambda \cdot \frac{\partial \rho}{\partial c} \right] + (1 - \lambda) \frac{\partial [E(r_1 \mid r_1 > \rho)]}{\partial c} - \frac{\partial \lambda}{\partial c} \cdot E(r_1 \mid r_1 > \rho)$$

It is not clear, however, which is the sign of this expression. In order to determine it, the expression must be further worked out.

(Step 1.1: $E(r_t \mid r_t \in [a, b]) = \int_{b}^{a} r_t \cdot g(r_t) dr_t \, ; \, a, b \in [0, R] ; \, a < b$)

Proof. Let $\theta = Pr(r_t \in [a, b]) = G(b) - G(a)$ and let $z$ be a continuous random variable, with support $[a, b] \subset [0, R]$, $a < b$, and pdf $f(z)$, such that the events $z \in [a', b']$ and $r_t \in [a', b']$ are equivalent, that is,

$$Pr(z \in [a', b']) = \tilde{\theta} \cdot Pr(r_t \in [a', b']) \, ,$$

for all $a', b' \in [a, b]$ with $a' \leq b'$, and $\tilde{\theta} \in \mathbb{R}$. This can be written as

$$Pr(z \in [a', b']) = \tilde{\theta} \cdot \int_{a'}^{b'} g(r_t) dr_t$$

Note that, in particular,

$$Pr(z \in [a, b]) = \tilde{\theta} \cdot \int_{a}^{b} g(r_t) dr_t = 1,$$

i.e.,

$$\tilde{\theta} \cdot \theta = 1,$$

or,

$$\tilde{\theta} = \frac{1}{\theta}$$

Then, we have

$$Pr(z \in [a', b']) = \frac{1}{\tilde{\theta}} \cdot Pr(r_t \in [a', b']) \, ,$$

i.e.,

$$\int_{a'}^{b'} f(z) dz = \frac{1}{\tilde{\theta}} \cdot \int_{a'}^{b'} g(r_t) dr_t,$$
or,

\[ \int_{a'}^{b'} f(z) \, dz = \int_{a'}^{b'} \frac{1}{\theta} \cdot g(r_t) \, dr_t, \quad (49) \]

for all \( a', b' \in [a, b] \) with \( a' \leq b' \). Then,

\[ f(z) = \frac{g(r_t)}{\theta} \quad (50) \]

As, for all \( a', b' \in [a, b] \) with \( a' \leq b' \), \( z \in [a', b'] \) and \( r_t \in [a', b'] \) are equivalent events, we have

\[ E(z \mid z \in [a', b']) = E(r_t \mid r_t \in [a', b']) \quad (51) \]

In particular,

\[ E(z \mid z \in [a, b]) = E(r_t \mid r_t \in [a, b]), \quad (52) \]

i.e.,

\[ E(z) = E(r_t \mid r_t \in [a, b]) \quad (53) \]

However, as

\[ E(z) = \int_{a}^{b} z \cdot f(z) \, dz = \int_{a}^{b} \frac{1}{\theta} \cdot r_t \cdot g(r_t) \, dr_t, \quad (54) \]

we have

\[ E(r_t \mid r_t \in [a, b]) = \frac{\int_{a}^{b} r_t \cdot g(r_t) \, dr_t}{\theta} \quad (55) \]

As \( \theta = G(b) - G(a) \), the result is established.

\[ (\text{Step 1.2:} \quad \int_{a}^{b} r_t g(r_t) \, dr_t = [r_t G(r_t)]_{a}^{b} - \int_{a}^{b} G(r) \, dr) \]

Proof. The result follows directly using integration by parts.
Now, using Steps 1.1-1.2, it is straightforward to show that $k$ can be expressed as

$$k = R - \int_{\rho}^{R} G(r_1) \, dr_1$$  \hspace{1cm} (56)$$

Then,

$$\frac{\partial k}{\partial c} = -\frac{\partial}{\partial c} \left[ \int_{\rho}^{R} G(r_1) \, dr_1 \right] = \int_{\rho}^{\rho + \frac{\partial \rho}{\partial c}} G(r_1) \, dr_1 > 0,$$  \hspace{1cm} (57)$$
as $\frac{\partial \rho}{\partial c} > 0$. \hfill \Box

(Step 2: $\frac{\partial b_1^*}{\partial c} < 0$)

Proof. We know that $b_1^* = \sqrt{\frac{c}{\xi}} - 1$. Then,

$$\frac{\partial b_1^*}{\partial c} = \frac{1}{2 \cdot \sqrt[3]{c}} \cdot \frac{\partial}{\partial c} \left[ \frac{k}{c} \right] = \frac{\sqrt{\xi} \cdot c - k}{2 \sqrt{k} \cdot c^2} = \frac{1}{2 \sqrt{k} \cdot c^2} \left[ \frac{\partial k}{\partial c} \cdot c - k \right]$$  \hspace{1cm} (58)$$

As $2 \sqrt{k} \cdot c^3 > 0$, the sign of $\frac{\partial b_1^*}{\partial c}$ will be determined by the sign of $\frac{\partial k}{\partial c} \cdot c - k$. As $\frac{k}{c} \cdot c - k = 0$, we will proceed by comparing $\frac{\partial k}{\partial c}$ to $\frac{k}{c}$.

(Step 2.1: $\frac{\partial k}{\partial c} < \frac{\mu}{c}$)

Proof. We know that

$$\frac{\partial k}{\partial c} = -\frac{\partial}{\partial c} \left[ \int_{\rho}^{R} G(r_1) \, dr_1 \right] = \int_{\rho}^{\rho + \frac{\partial \rho}{\partial c}} G(r_1) \, dr_1$$  \hspace{1cm} (59)$$

But,

$$\int_{\rho}^{\rho + \frac{\partial \rho}{\partial c}} G(r_1) \, dr_1 < \int_{\rho}^{\rho + \frac{\partial \rho}{\partial c}} 1 \, dr_1 = \rho + \frac{\partial \rho}{\partial c} - \rho = \frac{\partial \rho}{\partial c}$$  \hspace{1cm} (60)$$

Furthermore,

$$\frac{\partial \rho}{\partial c} = \beta \left( 1 - \frac{\sqrt{\eta}}{\sqrt[3]{\beta (\mu - \alpha)}} \right) \left( \frac{\sqrt{\mu}}{\sqrt[c]{c}} - 1 \right) < \frac{\mu}{c}$$  \hspace{1cm} (61)$$
Then,
\[ \frac{\partial k}{\partial c} < \frac{\mu}{c} \] (62)

(Step 2.2: \( \mu < k \))

Proof. It was already shown that \( k \) can be written as \( k = \lambda \rho + \int_0^\rho r_1 g( r_1 ) \, dr_1 \).

Note that
\[ \lambda \cdot E( r_1 | r_1 < \rho ) + \int_0^\rho r_1 g( r_1 ) \, dr_1 = \lambda \cdot \int_0^\rho r_1 g( r_1 ) \, dr_1 + \int_0^\rho r_1 g( r_1 ) \, dr_1 = \int_0^\rho r_1 g( r_1 ) \, dr_1 + \int_\rho^\infty r_1 g( r_1 ) \, dr_1 = \mu \] (63)

As \( E( r_1 | r_1 < \rho ) < \rho \), we have \( \mu < k \).  \( \square \)

Now, using Steps 2.1-2.2,
\[ \frac{\partial k}{\partial c} < \frac{\mu}{c} < \frac{k}{c} \] (64)

Then,
\[ \frac{\partial k}{\partial c} < \frac{k}{c} \] (65)

Therefore,
\[ \frac{\partial k}{\partial c} \cdot c - k < 0 \] (66)

and
\[ \frac{\partial b^*_1}{\partial c} < 0 \] (67)

\( \square \)

Now, as \( \pi^*_1 = \frac{1}{ b^*_1 + 1 } \), we have
\[ \frac{\partial \pi^*_1}{\partial c} = \frac{\partial \pi^*_1}{\partial b^*_1} \cdot \frac{\partial b^*_1}{\partial c} = -\frac{1}{(b^*_1 + 1)^2} \cdot \frac{\partial b^*_1}{\partial c} > 0 \] (68)

24
Then,

\[
\frac{\partial (1 - \pi^*_1)}{\partial c} < 0
\]  

(69)

\[
\square
\]

5.3 Appendix 3

(Part (i))

Proof. As \( b^*_2 = \frac{\sqrt{\beta}}{\sqrt{\eta}} - 1 \), it is clear that \( \frac{\partial b^*_2}{\partial \eta} = 0 \). Then, as \( \pi^*_2 = \frac{1}{b^*_2+1} \), we have

\[
\frac{\partial \pi^*_2}{\partial \eta} = \frac{\partial \pi^*_2}{\partial b^*_2} \cdot \frac{\partial b^*_2}{\partial \eta} = 0
\]  

(70)

Consequently,

\[
\frac{\partial (1 - \pi^*_2)}{\partial \eta} = -\frac{\partial \pi^*_2}{\partial \eta} = 0
\]  

(71)

\[
\square
\]

(Part (ii))

Proof. As \( m^* = \frac{\sqrt{\beta(\mu - \alpha)}}{\sqrt{\eta}} - 1 \), we have

\[
\frac{\partial m^*}{\partial \eta} = -\frac{\sqrt{\beta(\mu - \alpha)}}{2\eta^{12}} < 0
\]  

(72)

Then, as \( \tau^* = \frac{1}{1 + m^*} \), it follows that

\[
\frac{\partial \tau^*}{\partial \eta} = \frac{\partial \tau^*}{\partial m^*} \cdot \frac{\partial m^*}{\partial \eta} = -\frac{1}{(1 + m^*)^2} \left( -\frac{\sqrt{\beta(\mu - \alpha)}}{2\eta^{12}} \right) \\
= \frac{\eta}{\beta(\mu - \alpha)} \cdot \frac{\sqrt{\beta(\mu - \alpha)}}{2\eta^{12}} = -\frac{1}{2\eta^{12}} \cdot \frac{\sqrt{\beta(\mu - \alpha)}}{\sqrt{\eta}} > 0
\]  

(73)

\[
\square
\]

(Part (iii))
Proof. As \( \lambda = G(\rho) \), in order to determine the sign of \( \frac{\partial \lambda}{\partial \eta} \), it is first necessary to study \( \frac{\partial \rho}{\partial \eta} \). Once \( \rho \) can be expressed as

\[
\rho = \sqrt{\frac{\beta(\mu - \alpha)}{\sqrt{\eta}}} - \sqrt{\eta},
\]

we have

\[
\frac{\partial \rho}{\partial \eta} = 2 \left( \sqrt{\frac{\beta(\mu - \alpha)}{\sqrt{\eta}}} - \frac{1}{2\sqrt{\eta}} \right) = - \left( \frac{\sqrt{\beta(\mu - \alpha)}}{\sqrt{\eta}} - 1 \right) = -m^* < 0 \quad (74)
\]

Then, as

\[
\lambda = G(\rho) = \int_0^\rho g(r_1) \, dr_1, \quad (75)
\]

it follows that

\[
\frac{\partial \lambda}{\partial \eta} = \frac{\partial}{\partial \eta} \left[ \int_0^\rho g(r_1) \, dr_1 \right] = - \int_{\rho - m^*}^{\rho} g(r_1) \, dr_1 = - [G(\rho) - G(\rho - m^*)] < 0 \quad (76)
\]

(Step 1: \( \frac{\partial k}{\partial \eta} < 0 \))

Proof. It was already established that \( k \) can be expressed as \( k = R - \int_0^R G(r_1) \, dr_1 \). Then, as \( \frac{\partial k}{\partial \eta} = -m^* < 0 \), we have

\[
\frac{\partial k}{\partial \eta} = - \frac{\partial}{\partial \eta} \left[ \int_0^R G(r_1) \, dr_1 \right] = - \int_{\rho - m^*}^{\rho} G(r_1) \, dr_1 = - \int_{\rho - m^*}^{\rho} G(r_1) \, dr_1 < 0 \quad (77)
\]

(Step 2: \( \frac{\partial b^*_1}{\partial \eta} < 0 \))

Proof. As \( b^*_1 = \sqrt{\frac{k}{\pi}} - 1 \), we have

\[
\frac{\partial b^*_1}{\partial \eta} = \frac{\partial b^*_1}{\partial k} \cdot \frac{\partial k}{\partial \eta} = \frac{1}{2\sqrt{\pi k}} \cdot \left( - \int_{\rho - m^*}^{\rho} G(r_1) \, dr_1 \right) < 0 \quad (78)
\]

(Step 3: \( \frac{\partial (1 - \pi^*_1)}{\partial \eta} < 0 \))
Proof. As $\pi^*_1 = \frac{1}{\pi^*_1}$, we have

\[
\frac{\partial \pi^*_1}{\partial \eta} = \frac{\partial \pi^*_1}{\partial b^*_1} \cdot \frac{\partial b^*_1}{\partial \eta} = -\frac{1}{(b^*_1 + 1)^2} \left[ -\frac{1}{2\sqrt{k}} \int_{\rho-m}^{\rho} G(r_1) \, dr_1 \right]
\]

(79)

\[
= \frac{\sqrt{c}}{\sqrt{k}} \int_{\rho-m}^{\rho} G(r_1) \, dr_1 = \frac{\pi^*_1}{2k} \int_{\rho-m}^{\rho} G(r_1) \, dr_1 > 0
\]

Consequently,

\[
\frac{\partial (1 - \pi^*_1)}{\partial \eta} < 0 \tag{80}
\]

5.4 Appendix 4

Proof. Starting from $W(\lambda, \tau^*) = [\pi^*_1 + (1 - \pi^*_1) \lambda] \Delta + \beta [\pi^*_1 + (1 - \pi^*_1) \pi^*_2 [(1 - \lambda) + \lambda \tau^*]] \Delta$, it follows that

\[
\frac{\partial W(\lambda, \tau^*)}{\partial c} = \Delta \left[ \frac{\partial \pi^*_1}{\partial c} - \frac{\partial \pi^*_1}{\partial c} \lambda + (1 - \pi^*_1) \frac{\partial \lambda}{\partial c} \right]
+ \beta \Delta \left\{ \frac{\partial \pi^*_1}{\partial c} \cdot [\pi^*_1 + (1 - \pi^*_1) \frac{\partial \pi^*_2}{\partial c}] \cdot [(1 - \lambda) + \lambda \tau^*] \right\}
+ \beta \Delta \left\{ [(1 - \pi^*_1) \pi^*_2] \cdot \left[ \frac{\partial \lambda}{\partial c} + \frac{\partial \lambda}{\partial c} \tau^* + \lambda \frac{\partial \tau^*}{\partial c} \right] \right\}
\]

(81)

After some rearranging, this expression can be written as

\[
\frac{\partial W(\lambda, \tau^*)}{\partial c} = \Delta \left[ (1 - \pi^*_1) \cdot (1 - \beta (1 - \tau^*) \pi^*_2) \right]_{>0}
+ \Delta \left[ \frac{\partial \lambda}{\partial c} \cdot (1 - \pi^*_1) \cdot [1 - \beta (1 - \tau^*) \pi^*_2] \right]_{>0}
+ \Delta \left[ \frac{\partial \pi^*_2}{\partial c} \cdot \beta (1 - \pi^*_1) [(1 - \lambda) + \lambda \tau^*] + \Delta \frac{\partial \tau^*}{\partial c} \cdot \beta \lambda (1 - \pi^*_1) \pi^*_2 \right]_{<0}
\]

(82)

(\text{I})

(\text{II})

(\text{III})

(\text{IV})

Thus, in order to show that $\frac{\partial W(\lambda, \tau^*)}{\partial c} > 0$, it is sufficient to establish
that $(III) + (IV) > 0$. It follows that

$$(III) + (IV) = \beta (1 - \pi_2^*) \left\{ \frac{\partial \pi_2^*}{\partial c} [(1 - \lambda) + \lambda \tau^*] + \frac{\partial \tau^*}{\partial c} \lambda \pi_2^* \right\}$$

(83)

Substituting $\frac{\partial \pi_2^*}{\partial c}$ and $\frac{\partial \tau^*}{\partial c}$ by already known expressions leads to

$$(III) + (IV) = \beta (1 - \pi_1^*) \left\{ \frac{1}{2 \sqrt{\mu c}} [(1 - \lambda) + \lambda \tau^*] - \tau^* \frac{b_2^*}{2 (\mu - \alpha)} \lambda \pi_2^* \right\}$$

(84)

After further rearranging, this can be expressed as

$$(III) + (IV) = \beta (1 - \pi_1^*) \left\{ \frac{1 - \lambda}{\sqrt{\mu c}} + \lambda \tau^* \cdot \left[ \frac{1}{\sqrt{\mu c}} - \frac{(1 - \pi_2^*)}{(\mu - \alpha)} \right] \right\}$$

(85)

Finally, we show that $(*) > 0$:

(Step 1: $\sqrt{\mu c} < \mu - \alpha$)

**Proof.** It is straightforward that $\mu - \alpha = 2 \sqrt{\mu c} - c$. Then,

$$\mu - \alpha - \sqrt{\mu c} = 2 \sqrt{\mu c} - c - \sqrt{\mu c} = \sqrt{\mu c} - c > 0,$$

(86)

as $\mu > c$.

(Step 2: $(*) > 0$)

**Proof.** From the previous step, it follows that

$$\frac{1}{\sqrt{\mu c}} > \frac{1}{\mu - \alpha} > 0$$

(87)

Furthermore,

$$(*) = \frac{1}{\sqrt{\mu c}} - \frac{(1 - \pi_2^*)}{(\mu - \alpha)} > \frac{1}{\sqrt{\mu c}} - \frac{1}{\mu - \alpha} > 0$$

(88)

As $(*) > 0$, it follows that $(III) + (IV) > 0$ and $\frac{\partial W(\lambda, \tau^*)}{\partial c} > 0$. □
References


