Ideology or valence?
Redistributive policies in a model with asymmetric candidates and ideologically biased voters

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Abstract
This paper considers a model where two parties with distinct political ideology compete in a proportional election presenting two candidates with different valence. In this context, party redistributive policies are analyzed to identify the elements which favor the rise of a machine politics outcome or of a swing voter outcome. In particular, the role and the interactions between the distribution of ideology in the electorate and candidate valence are analyzed. JEL Classification: D72, D63, H2. Keywords: Ideology, valence, redistributive policies.
1 Introduction

When facing the decision how to vote, a person must evaluate the different proposals coming from the politicians running for office. These proposals typically consist of many elements bundled in the electoral platforms presented to the electorate.

The first element which catches the attention of the voter is probably the identity of the candidate who a party proposes for the office. Candidate characteristics and personal history define in fact the level of valence which we can roughly describe as the perceived ability to manage the public budget, and the propensity toward corruption and embezzlement. These aspects which are strongly affected by the effects of campaign spending, \(^1\) are usually summarized in the dead-weight loss associated with the redistributive policies proposed by each candidate \(^2\) and ultimately define the size of the budget available for redistribution.

A second aspect which is relevant to voter decision is the party backing the candidate. In this case, history and background of the party define a set of fixed positions on specific issues, whose nature may be economic, as in the case of the programmatic redistribution defined in Dixit and Londregan (1996), or non-economic as in the case of abortion and gun control, characterizing its broad political view, i.e. its ideology.

Both the valence of the candidate and the ideology of a party are issues over which politicians have limited control and are not easily changed. The third element included in a political proposal, i.e. redistributive policies or tactical redistribution using the definition of Dixit and Londregan (1996) (or again pork-barrel spending in a less noble interpretation), is instead the element which can be more freely steered by politicians and used to gather votes. In recent times, the role of tactical redistribution has dramatically increased following the development of micro-targeting techniques in campaign spending, which allow candidates to target their efforts toward each voter with individualized messages and promises. This has happened to the extent that Jamieson (2013) envisioned a change in the focus of electoral competition from swing states to swing individuals.

The influence on redistributive policies of a candidate valence and of party ideology has been studied by different authors in contexts where politicians know the characteristics of each voter in the electorate in terms of ideology and in terms of the level of valence accorded to a candidate, and where perfect targeting of redistribution is possible. In

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\(^1\) On this aspect, see, for instance, Sahuguet and Persico (2006)

\(^2\) On this interpretation in terms of the leaky bucket used for redistribution, see, for instance, Kovenock and Roberson (2009)
particular, Sahuguet and Persico (2006) and Kovenock and Roberson (2008) analyze the effects of valence on redistribution, while Magnani (2017) focuses on the effects of ideology. None of these analyses though, has ever considered the three elements involved in voter choice in the same setting nor has analyzed their interactions and mutual effects on redistributive policies.

The present paper performs this analysis and complements the previous literature. This is done by trying to answer the following questions: which are the voters who mostly benefit from redistribution? How is this set of privileged beneficiaries modified when the dead-weight loss associated with party redistributive policies changes and when the dispersion of ideology in the electorates varies? In order to provide an answer to these questions we classify each voter on the basis of the ideological bias toward a party, identifying the core supporters of each party and the swing voters, i.e. the voters who have no ideology bias, and fully characterize the equilibrium of the redistribution game. We find that the party with the smallest budget for redistribution adopts a set of policies which favors its core supporters, who receive the highest expected transfers, while voter who have an ideological bias for the opponent party receive less. The party with the largest budget instead, targets the highest expected transfers to the voters who are ideologically biased toward the opponent party at the expenses of its core supporters.

In this context, an increase in the dispersion of ideology in the electorate causes the party with the smallest budget to increase its focus on core supporters. The party with the largest budget, on the contrary, reduces expected transfers to all voters, and especially those targeted to voters who are biased toward the opponent.

We further analyze the effects of changes in the absolute size of party budget, and find that if the party with the smallest budget faces a contraction in its resources available for redistribution, it reacts reducing its expected transfers to all voters and decreasing particularly the size of those targeted to its core supporters. On the contrary, a reduction in the resources available to the party with the largest budget affects the redistributive policies of both parties, and has effects analogous to those caused by an increase in the dispersion of ideology.

The paper has the following structure. Section 2 reviews the related literature. Section 3 presents the model. Section 4 characterizes the equilibrium in the benchmark case of symmetric parties. Section 5 analyzes the case where the parties have different budgets available for redistribution. Section 6 duly concludes.
2 Related Literature

The present paper is based on a recent work by Magnani (2017) which studies the effects of ideology on the redistributive policies chosen by two symmetric parties which compete for office in a proportional election. In this context, voter preferences are linear in the transfers received through redistributive policies but also include an additive $x$ term that defines the ideology bias of each voter for a specific party. In equilibrium, party transfers decrease with the size of the ideology bias, and a “swing voter” outcome emerges.

We extend this framework by assuming that the parties present in the elections two candidates who have different levels of valence implying that, just as in Sahuguet and Persico (2006), each party (or equivalently, each candidate) has a different budget available for its redistributive policies. Introducing this new elements allows to study the interactions between valence and ideology and to characterize the determinants of party equilibrium redistributive policies.

The present setting where ideology is symmetrically distributed among voters and where parties/candidates have different valence is novel to the literature. The present analysis thus, complements the previous literature on deterministic voting that, starting from the seminal contribution by Myerson (1993), has studied the effects of political competition on a range of issues including public deficits (Lizzeri, 1999) inequality (Laslier, 2002; Laslier and Picard, 2002; Kovenock and Roberson, 2008), campaign spending regulation (Sahuguet and Persico, 2006), inefficient redistributive policies (Crutzen and Sahuguet, 2009, and Kovenock and Roberson, 2009) and the provision of pure (Lizzeri and Persico, 2001 and 2005) and local public goods (Magnani, 2009, 2010 and 2013).

The results of our analysis provide a new insight on the effects on redistributive policies of changes in the dispersion of voter ideology and of changes in the absolute size of party budgets available for redistribution. In particular, they permit to identify the factors which favor the rise of a swing voter outcome or of a machine politics outcome.

The issue of electoral competition with ideologically biased voters is also analyzed in a probabilistic voting setting in several papers (Lindbeck and Weibull 1987, Dixit and Londregan 1996, and more recently Krasa and Polborn 2014).

A common feature of these models is the fact that party offers are homogenous within each group while group members are heterogeneous. In the present setting instead, we allow for micro-targeting in campaign spending which is a phenomenon whose importance in electoral competition has recently increased as documented by Boyer and Konrad (2014). This implies that there are homogeneous groups in the electorate, be-
cause the parties can perfectly discriminate among voters, and target each of them with their redistributive policies.

This different point of view on electoral competition provides a novel insight on the effects of ideology and valence on the outcomes of redistributive policies, which complements the previous findings of the probabilistic voting literature.

3 The model

The model analyzes the competition between two symmetric parties running for office in a proportional election, and includes two stages. In the first stage, Party $A$ and Party $B$ simultaneously present electoral platforms to which they credibly commit. In the second stage, elections are held and voters cast their votes.

In the economy there is a continuum of citizens denoted by the interval $[0, 1]$. The parties compete over this electorate presenting their electoral platforms which include a redistribution program for the public budget. The public budget is initially assumed to be the same for both parties, and has a measure equal to 1. The redistribution plan defines a non-negative offer for each person that turn out to vote, and specifies, for generic voter $v$, the quantity of a homogeneous good which Party $i$ ($i = \{A, B\}$) offers her, $x_v^i(v) \in [0, +\infty)$. Party redistributive politics must fulfill a balanced budget constraint which requires that offers to voters do not average more than the size of the public budget, 1.

The parties are opportunistic, and allocate public money in order to maximize vote shares, $S_A$ and $S_B$. This allows them to maximize their influence over the political process. After the elections indeed, the probability that Party $i$’s platform is implemented is an increasing function of $S_i$.4

In the choice of their redistributive politics, the parties must consider the fact that the population is not homogeneous. In particular, citizens have fixed preferences for the parties whose strength is summarized by the ideological bias $\eta$ which defines the appreciation for the ideology supported by Party $A$. The ideology bias enters additively in the utility function. As a consequence, a citizen who turns out to vote and receives an electoral promise $x$ from Party $A$ obtains

$$u(x) = x + \eta.$$ 

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3Suppose for instance, that each citizen pays a lump-sum tax of this amount.

4We are assuming that the process leading to the definition of post-elections politics is a probabilistic compromise of the type described by Sahuguet and Persico (2006).
If $\eta > 0$ the citizen is a partisan (core supporter) of Party $A$; if $\eta < 0$ the citizen is a partisan (core supporter) of Party $B$, and if $\eta = 0$ the citizen has no ideological preferences and is a swing voter.\(^5\) The ideological bias $\eta$ is distributed in the population according to the cumulative distribution function $\Phi(\eta)$ on the support $[-\bar{\eta}, \bar{\eta}]$, with a density function $\phi(\eta)$. The parties observe voter ideological preferences implying that a strategy for Party $i$, $\Psi_i$, is a function linking each voter $v$, with a generic ideology bias $\eta$, to an offer of a share of its budget $x^i_v(\eta) \in [0, +\infty)$ i.e.:

$$\Psi^i(v, \eta) : [0, 1] \times [0, +\infty) \rightarrow x^i_v(\eta)$$

where $x^i_v(\eta)$ fulfills the budget constraint

$$\int_0^1 x^i_v(\eta) \cdot dv \leq 1.$$ 

When elections are held, votes are cast simultaneously. People vote sincerely, and choose between Party $A$ and Party $B$ on the basis of which offer provides the highest utility. A strategy for the generic voter $v$, $\Psi_v$, is a function of party offers

$$\Psi_v(x^A_v(\eta); x^B_v(\eta)) = [0, +\infty) \times [0, +\infty) \rightarrow \{A, B\}.$$ 

In case a tie occurs because party offers ensure the same utility level, the voter chooses each party with the same probability.

4 Symmetric parties

We now characterize the equilibrium for the setting where the parties are symmetric. The game is sequential and is solved by backward induction. In the last stage, the elections stage, voters choose, in equilibrium, the party which offers them the highest utility. In the first stage, where party choose the redistributive politics to include in their electoral platforms, the analysis of the equilibrium is usefully performed adopting the approach developed by Sahuguet and Persico (2006). This approach considers the maximization problem of Party $i$ and establishes the strategic equivalence with the maximization problem of a player who competes with an opponent in an infinite number of independent all-pay auctions, each corresponding to a specific point in the distribution of $\eta$. The equivalence, which is proven in the Appendix, allows to exploit several results from the all-pay auction literature, and in the first place, allows to say

\(^5\)As in Krasa and Polborn (2014), this characterization of voter preferences approximates a more complex setting, where an electoral platform includes a set of "core" policies, which are shaped by ideology and thus fixed, and a set of adjustable pledge policies, which define the redistributive policies of a party.
that only mixed strategy equilibria exists for the redistribution game. In analyzing party strategies thus, we focus on the cumulative probability function $F_i(x^i)$, which characterizes the distribution of Party $i$’s electoral promise to each voter with an ideological bias equal to $\eta$. In particular, Party $i$ makes the stochastic offer, $x^i_\eta$, which is an independent draw from $F_i^j(x^i_\eta)$. Thus party redistribution plans are an infinite sequence of independent draws.

Note now that the density function $\phi(\eta)$ defines the dimension of the segment of the infinite electorate which includes the voters who have an ideology bias equal to $\eta$. This segment includes a continuum of voters implying that the law of large numbers applies here. This means that $F_i^j(x^i_\eta)$ also defines the share of voters with an ideology bias $\eta$ who receive an offer lower than $x^i_\eta$.

Given these remarks, Party $i$’s maximization problem is

$$
Max_{x^i_\eta} \int_{-\eta}^{\eta} \phi(\eta) \int_0^\infty F_i^j(x^i_\eta + \eta) \cdot dF_i^j(x^i_\eta) \cdot d\eta
$$

$$
+ \lambda \left[ 1 - \int_{-\eta}^{\eta} \phi(\eta) \left( \int_0^\infty x^i_\eta \cdot dF_i^j(x^i_\eta) \right) \right].
$$

The equivalence result makes it possible to break up party strategies, and separately analyze the redistributive policies targeted to voters with a specific ideology bias $\eta$. In equilibrium, a crucial role in the definition of these policies is played by the comparison between $\eta$ and $\frac{1}{\lambda}$.

In this framework, the optimal choice of the Lagrangian multiplier attached to party budget constraint, $\lambda$, defines party shadow cost of money. The inverse of this quantity, $\frac{1}{\lambda}$, defines the relative price of one vote in terms of money, given by the ratio between the value of one vote and the value of one unit of money. In other words, $\frac{1}{\lambda}$ defines the monetary appraisal which the parties make of each vote.

Note now that competing for a voter with an ideological bias $\eta < 0$ requires Party $A$ to pay the entry cost $|\eta|$. A transfer of at least $|\eta|$, in fact, is needed to provide the voter with a utility level at least equal to that provided by ideology, and to have a positive probability of obtaining his/her vote. The appraisal which Party $A$ makes of these vote, net of the entry cost, is thus $\frac{1}{\lambda} - |\eta|$, due to the effects of the ideology bias.

\[6 ^{6}\text{Clearly, it is always optimal for Party } i \text{ to make the same offer to all the voters with the same ideology bias.} \]

\[7 ^{7}\text{Both the application of the law of large number and the independence assumption for the random draws pose problems, which are discussed by Myerson (1993) and Alos-Ferrer (2002). Interpreting the continuum of voters as an approximation for a large finite number of voters overcomes these technical difficulties.}\]
In this context, the difference between the entry cost, $|\eta|$, and $\frac{1}{\lambda}$ defines the intensity of the competition for a voter with an ideological bias $\eta < 0$. The closer (further) is this entry cost to the appraisal which Party $A$ makes of these vote, the lower (higher) competition is going to be. It is worth noting that a voter with an ideology bias $\eta < 0$, such that $|\eta| > \frac{1}{\lambda}$, does not receive any offer from Party $A$. In this case indeed, the entry cost exceeds the relative price of the vote in terms of money, hence defining a negative net appraisal.

A symmetrical argument also holds for Party $B$ when the ideological bias is $\eta > 0$. In the present analysis, we focus on the circumstance where all voters receive the promise of a transfer from both parties. This requires that that $\frac{1}{\lambda} \geq |\eta|$ always holds; a sufficient condition for this inequality to be satisfied is $|\bar{\eta}| < 2$, as we will show below.

Given these remarks, consider the equilibrium for the redistribution game.

**Theorem 1** A unique mixed-strategy Nash equilibrium for the redistribution game exists where party strategies and payoffs are characterized as follows.

If $\eta \leq 0$, Party $A$ makes a nil offer with probability $\lambda \cdot |\eta|$, and, with complementary probability, randomizes according to the cumulative distribution function

$$F^A\left(x^A\right) = \lambda \cdot x^A$$

over the support $\left(|\eta|, \frac{1}{\lambda}\right]$.

Party $B$ makes a nil offer with probability $\lambda \cdot |\eta|$, and, with complementary probability, randomizes according to the cumulative distribution function

$$F^B\left(x^B\right) = \lambda \left(x^B + |\eta|\right)$$

over the support $\left(0, \frac{1}{\lambda} - |\eta|\right]$.

If $\eta > 0$, party strategies are symmetric.

The unique value of $\frac{1}{\lambda}$ which solves the maximization problem of the parties is:

$$\frac{1}{\lambda} = 2 + mad[\eta].$$

where $mad[\eta] = \int_{-\eta}^{0} \phi(\eta)|\eta| \cdot d\eta = 2 \cdot \int_{0}^{\eta} \phi(\eta)\eta \cdot d\eta$ is the mean absolute deviation around the mean.

Party vote shares are $S^A = S^B = \frac{1}{2}$.

**Proof.** See the Appendix. ■

Note that party shadow cost of money is such that $\frac{1}{\lambda} \geq 2$ implying further that, by our previous assumption, $\frac{1}{\lambda} \geq |\bar{\eta}|$ holds.
By the previous characterization of party equilibrium strategies, it is possible to define the size and the distribution of expected party transfer among voters.

**Corollary 2** In equilibrium, the expected transfers of Party $i$ ($i = A, B$) are characterized as follows:

- $E \left[ x_{\eta}^i \right] = \frac{1}{2} \cdot \frac{1}{\lambda} - \frac{1}{2} \left( |\eta| \right)^2$ for the voters who are ideologically biased toward the opponent;

- $E \left[ x_{\eta}^i \right] = \frac{1}{2} \cdot \frac{1}{\lambda} + \frac{1}{2} \left( |\eta| \right)^2 - |\eta|$ for Party $i$'s partisan voters

Hence for given absolute value of $\eta$, the core supporters of a party receive an expected transfer which is lower than that received by voters who are biased toward the opponent.

Expected transfers are decreasing in the size (absolute value) of $\eta$, implying that swing voters are those who receive the highest expected transfers.

**Proof.** From Theorem 1 straightforwardly follows that, in equilibrium, a voter with the ideology bias $\eta < 0$ receives the following expected transfers:

$$E \left[ x_{\eta}^A \right] = \int_{|\eta|}^{\frac{1}{2}} \lambda \cdot x_{\eta}^A \cdot dx_{\eta} = \frac{1}{2} \cdot \frac{1}{\lambda} - \frac{1}{2} \left( |\eta| \right)^2$$

(5)

from Party $A$, and

$$E \left[ x_{\eta}^B \right] = \int_{0}^{\frac{1}{2}-|\eta|} \lambda \cdot x_{\eta}^B \cdot dx_{\eta} = \frac{1}{2} \cdot \frac{1}{\lambda} + \frac{1}{2} \left( |\eta| \right)^2 - |\eta|$$

(6)

from Party $B$. The reverse clearly holds if $\eta > 0$.

Compare now $E \left[ x_{\eta}^A \right]$ and $E \left[ x_{\eta}^B \right]$ and note that

$$\frac{1}{2} \cdot \frac{1}{\lambda} - \frac{1}{2} \left( |\eta| \right)^2 - \frac{1}{2} \cdot \frac{1}{\lambda} - \frac{1}{2} \left( |\eta| \right)^2 + |\eta| = |\eta| - \lambda \left( |\eta| \right)^2 > 0$$

(7)

for every $|\eta|$ by the assumption $\frac{1}{2} \geq |\eta|$.

Note now that

$$\frac{\partial E \left[ x_{\eta}^A \right]}{\partial |\eta|} = -\lambda |\eta| < 0$$

(9)

and

$$\frac{\partial E \left[ x_{\eta}^A \right]}{\partial |\eta|} = -1 + \lambda |\eta| < 0$$

(10)
by the assumption $\frac{1}{\lambda} \geq \eta$. ■

This description of the effects of ideology conforms to the "swing voter" theory, and is coherent with the results obtained by Magnani (2017) in a setting where ideology is a discrete variable. This means that swing voters, i.e. voters whose ideology bias is $\eta = 0$ receive the largest expected transfer from both parties. At the same time voters whose ideology bias is $\eta < 0$ and are partisan of Party B receive from Party A a larger expected transfer than voters who are partisan of Party A and have an ideology bias equal to $-\eta > 0$. Symmetric results characterize the redistributive policies chosen by Party B.

Consider now the effects of a variation in the dispersion of ideology on party redistributive policies and define first $E^i_{-}[x^i_{\eta}]$ and $E^i_{+}[x^i_{\eta}]$ with $i = A, B$ as the value of the expected transfer offered by Party $i$ to a voter with an ideology bias respectively $\eta \leq 0$ and $\eta > 0$.

**Corollary 3** An increase in the mean average deviation of ideology distribution causes

- an increase in party expected transfers to all voters, with a larger increase for voters who are ideologically biased toward the opponent;

- an increase in the degree of inequality of redistributive policies.

**Proof.** Consider Equation (4) and note that

$$\frac{\delta 1}{\delta \text{mad}[\eta]} = 1. \quad (11)$$

This implies that if $\eta < 0$:

$$\frac{\delta E^A_{-}[x^A_{\eta}]}{\delta \text{mad}[\eta]} = \frac{1}{2} \left[1 + (\eta \cdot \lambda)^2\right] > 0 \quad (12)$$

and when $\eta \geq 0$

$$\frac{\delta E^A_{+}[x^A_{\eta}]}{\delta \text{mad}[\eta]} = \frac{1}{2} (1 + \lambda \cdot \eta) (1 - \lambda \cdot \eta) > 0 \quad (13)$$

holds since $\frac{1}{\lambda} \geq \eta$. Note now that for every pair of values $\eta_+ \geq 0$ and $\eta_- < 0$ it is always the case that $\frac{\delta E^A[\eta^A]}{\delta \text{mad}[\eta]} > \frac{\delta E^A[\eta^A]}{\delta \text{mad}[\eta]}$ since

$$\frac{1}{2} \left[1 + (\eta_- \cdot \lambda)^2\right] - \frac{1}{2} \left[1 + (\eta_+ \cdot \lambda)^2\right] = (\eta_- \cdot \lambda)^2 + (\eta_+ \cdot \lambda)^2 > 0. \quad (14)$$
From previous results straightforwardly follows that in equilibrium, the redistributive policies targeted to voter with a generic ideological bias $\eta$, which are presented before an increase in ideology dispersion, are second-order stochastically dominated by those presented after this event has occurred. This is clearly shown by the fact that the upper bound in the support of the distribution increases ($\frac{1}{\lambda}$), and, at the same time, the the probability attached to every transfer decreases ($\lambda$). The reverse occurs when ideology dispersion decreases.

An increase in ideology dispersion causes an increase in the size of the groups which, having a larger ideology bias, are less "costly" to contest from the party point of view since it targets a low level of expected transfers to these voters. This results in a saving of resources which are used to increase the expected transfers of all voters and specially those targeted to people who are ideologically biased toward the opponent party.

5 Asymmetric parties

In this section we introduce an asymmetry in the resources available to the parties. This allows us to analyze its effects on redistributive policies.

In order to do that, we consider a setting where each party has a different valence, interpreted as the size of the deadweight loss involved in redistribution. In particular, we assume that Party $A$ is perceived as being less valent than Party $B$, implying that the deadweight loss in its redistributive policies is larger than that incurred by Party $B$. We model this circumstances by denoting by $\beta^i$ the deadweight loss in Party $i$’s redistributive policies, so that its budget available for redistribution amounts to $1 - \beta^i$, and by assuming $\beta^A > \beta^B$. In this context, Party $i$’s transfer targeted to voter $v$ is given by $x^i_\eta (1 - \beta^i)$.

We further assume for model tractability, that Party $A$ is the disadvantaged party in all the segments of the electorate. This is equivalent to say that party shadow costs of money, $\lambda^A$ and $\lambda^B$ are such that $\frac{1}{\lambda^A} - \frac{1}{\lambda^B} \geq \bar{\eta}$.

In this context, Party $i$’s maximization problem becomes

$$Max_{F^i_\eta(x^i_\eta)} \int_{-\bar{\eta}}^{\bar{\eta}} \phi(\eta) \int_0^\infty F^j_\eta \left( \frac{1 - \beta^i}{1 - \beta^j} \cdot x^i_\eta + \frac{\eta}{1 - \beta^j} \right) \cdot dF^i_\eta (x^i_\eta) \cdot d\eta +$$

$$+ \lambda^i \left[ 1 - \int_{-\bar{\eta}}^{\bar{\eta}} \phi(\eta) \left( \int_0^\infty x^i_\eta \cdot dF^i_\eta (x^i_\eta) \right) d\eta \right] (15)$$

We now characterize the equilibrium for the redistribution game. Define initially $\rho = \frac{1 - \beta^A}{1 - \beta^B} \leq 1$, and exploit the equivalence result proved
in the Appendix. Following the same steps described in the previous section, we can state that:

**Theorem 4** A unique mixed-strategy Nash equilibrium for the redistribution game exists where party strategies and payoffs are characterized as follows.

If \( \eta \leq 0 \), Party A randomizes according to the cumulative distribution function

\[
F^A_\eta(x^A_\eta) = 1 - \frac{\lambda^B}{\lambda^A} + \rho \cdot \lambda^B \cdot x^A_\eta
\]

over the support \( \left(-\frac{\eta}{1-\beta^A}, \frac{1}{\rho} \cdot \frac{1}{\lambda^A}\right) \).

Party B randomizes according to the cumulative distribution function

\[
F^B_\eta(x^B_\eta) = \lambda^A \left(x^B_\eta - \frac{\eta}{1-\beta^B}\right)
\]

over the support \( \left(0, \frac{1}{\lambda^A} + \frac{\eta}{1-\beta^A}\right) \).

If \( \eta > 0 \), Party A randomizes according to the cumulative distribution function

\[
F^A_\eta(x^A_\eta) = 1 - \frac{\lambda^B}{\lambda^A} + \rho \cdot \lambda^B \cdot x^A_\eta
\]

over the support \( \left(0, \frac{1}{\rho} \cdot \frac{1}{\lambda^A}\right) \).

Party B randomizes according to the cumulative distribution function

\[
F^B_\eta(x^B_\eta) = \lambda^A \left(x^B_\eta - \frac{\eta}{1-\beta^B}\right)
\]

over the support \( \left(-\frac{\eta}{1-\beta^A}, \frac{1}{\lambda^A} + \frac{\eta}{1-\beta^A}\right) \).

The unique values for the inverse of the Lagrangean multipliers which solve the maximization problem of the parties are:

\[
\frac{1}{\lambda^A} = 1 + \sqrt{1 + \frac{1}{2} \cdot \frac{\text{var}[\eta]}{(1-\beta^B)^2}}
\]

and

\[
\frac{1}{\lambda^B} = \frac{1}{\rho} \cdot \frac{1}{\lambda^A} \cdot \sqrt{1 + \frac{1}{2} \cdot \frac{\text{var}[\eta]}{(1-\beta^B)^2}}
\]

Party vote shares are

\[
S^A = \frac{1}{2} \cdot \frac{\lambda^B}{\lambda^A} + \frac{1}{2} \cdot \frac{\lambda^B}{1-\beta^B} \int_{-\eta}^{0} \phi(\eta)\eta d\eta
\]
and
\[ S^B = 1 - \frac{1}{2} \cdot \frac{\lambda^B}{\lambda^A} + \frac{1}{2} \cdot \frac{\lambda^B}{1 - \beta^B} \int_{-\bar{\eta}}^0 \phi(\eta)\eta d\eta. \] (23)

**Proof.** See the Appendix. ■

Note that, as expected, it is:
\[ \lim_{\text{var}[\eta] \to 0} \frac{1}{\lambda^A} = 2 \]
and
\[ \lim_{\text{var}[\eta] \to 0} \frac{1}{\lambda^B} = \frac{2}{\rho}. \]

Reconsider now our initial assumption \( \frac{\lambda^A}{\lambda^B} - \frac{1}{\lambda^A} \geq \bar{\eta} \) and note that by Equations (20) and (21) it is always possible to find a \( \beta^A \) sufficiently large for the previous condition to be satisfied. 8

Consider now party expected transfers.

**Corollary 5** In equilibrium, the expected transfers of Party A are characterized as follows:

\[ E^A_+ [x^A_\eta] (1 - \beta^A) = \frac{1 - \beta^A}{2} \cdot \frac{1}{\lambda^A} \left[ 1 + \frac{1}{2} \cdot \frac{\text{var}[\eta]}{(1 - \beta^B)^2} \right]^{-\frac{1}{2}} \left[ 1 - \left( \frac{\eta \cdot \lambda^A}{1 - \beta^B} \right)^2 \right] \] (24)

and

\[ E^A_- [x^A_\eta] (1 - \beta^A) = \frac{1 - \beta^A}{2} \cdot \frac{1}{\lambda^A} \left[ 1 + \frac{1}{2} \cdot \frac{\text{var}[\eta]}{(1 - \beta^B)^2} \right]^{-\frac{1}{2}}. \] (25)

Party A’s expected transfers weakly increase in \( \eta \).

This implies that partisan voters are offered an expected transfer that, for every value of \( \eta \), is larger than that offered to voters who are biased toward the opponent.

In the case of Party B’s redistributive policies the expected transfers offered are characterized as follows:

\[ E^B_- [x^B_\eta] (1 - \beta^B) = \frac{1 - \beta^B}{2} \cdot \frac{1}{\lambda^A} \left[ 1 + \left( \frac{\lambda^A \cdot \eta}{1 - \beta^B} \right)^2 \right] + \eta \] (26)

and

\[ E^B_+ [x^B_\eta] = \frac{1 - \beta^B}{2} \cdot \frac{1}{\lambda^A} + \eta. \] (27)

Party B’s expected transfer are strictly increasing in \( \eta \).

This implies that partisan voters are offered an expected transfer that, for every value of \( \eta \), is smaller than that offered to voters who are biased toward the opponent.

8Note indeed that \( \lim_{\beta^A \to 1} \frac{1}{\lambda^A} = +\infty \)
Proof. Consider initially the expected transfer of Party $A$ to voters who are ideologically biased toward the opponent, so that $\eta \leq 0$. In this case, it is

$$E^A_\eta \left( x^A \right) \left( 1 - \beta^A \right) = \frac{\lambda^B}{2} \cdot \frac{1 - \beta^A}{\rho} \left( \frac{1}{\lambda^A} \right)^2 - \frac{\rho \cdot \lambda^B}{2} \left( \frac{\eta}{1 - \beta^A} \right)^2 \left( 1 - \beta^A \right)$$

or substituting $\frac{1}{\chi^2}$, simplifying and reordering the terms

$$E^A_\eta \left( x^A \right) \left( 1 - \beta^A \right) = \frac{1 - \beta^A}{2} \cdot \frac{1}{\lambda^A} \left[ 1 + \frac{1}{2} \cdot \frac{\text{var}[\eta]}{(1 - \beta^B)^2} \right]^{-\frac{1}{2}} \left[ 1 - \left( \frac{\eta \cdot \lambda^A}{1 - \beta^B} \right)^2 \right].$$

In case $\eta > 0$

$$E^A_\eta \left( x^A \right) \left( 1 - \beta^A \right) = \frac{\rho \cdot \lambda^B}{2} \left( \frac{1}{\lambda^A} \cdot \frac{1}{\lambda^A} \right)^2 \left( 1 - \beta^A \right)$$

or substituting $\frac{1}{\chi^2}$, simplifying and reordering the terms

$$E^A_\eta \left( x^A \right) = \frac{1 - \beta^A}{2} \cdot \frac{1}{\lambda^A} \left[ 1 + \frac{1}{2} \cdot \frac{\text{var}[\eta]}{(1 - \beta^B)^2} \right]^{-\frac{1}{2}}.$$

Observe now that clearly $E^A_\eta \left( x^A \right) \left( 1 - \beta^A \right) \geq E^A_\eta \left( x^A \right) \left( 1 - \beta^A \right)$ holds.

Moreover it is the case that

$$\frac{\partial E^A_\eta \left( x^A \right) \left( 1 - \beta^A \right)}{\partial \eta} = -\frac{\eta}{\lambda^A} \left[ 1 + \frac{1}{2} \cdot \frac{\text{var}[\eta]}{(1 - \beta^B)^2} \right]^{-\frac{1}{2}} \left( \frac{\lambda^A}{1 - \beta^B} \right)^2 > 0$$

and

$$\frac{\partial E^A_\eta \left( x^A \right)}{\partial |\eta|} = 0$$

implying that Party $A$’s expected transfers to the voters are weakly increasing with the value of the ideology bias.

Consider now Party $B$’s expected transfers when $\eta \leq 0$

$$E^B_\eta \left( x^B \right) \left( 1 - \beta^B \right) = \frac{1 - \beta^B}{2} \cdot \frac{1}{\lambda^A} + \eta \left( 1 + \frac{\lambda^A}{2} \cdot \frac{\eta}{1 - \beta^B} \right).$$

If instead it is $\eta > 0$, Party $B$’s expected transfer is

$$E^B_\eta \left( x^B \right) \left( 1 - \beta^B \right) = \frac{1 - \beta^B}{2} \cdot \frac{1}{\lambda^A} + \eta.$$
Note now that since
\[
\frac{1}{2} \cdot \lambda^A + \left( \frac{\eta}{1 - \beta^B} \right) \geq \frac{1}{2} \cdot \lambda^A - \left( \frac{|\eta|}{1 - \beta^B} \right) \left( 1 + \frac{\lambda^A}{2} \cdot \frac{\eta}{1 - \beta^B} \right) \tag{36}
\]
it is the case that \( E^B_+[x^B_\eta] (1 - \beta^B) \geq E^B_-[x^B_\eta] (1 - \beta^B) \forall \eta \).

Consider further that
\[
\frac{\partial E^B_-[x^B_\eta]}{\partial \eta} = \left( \frac{1}{1 - \beta^B} \right) \left[ 1 + \lambda^A \cdot \frac{\eta}{1 - \beta^B} \right] \geq 0 \tag{37}
\]
by the assumption \( \frac{1}{\lambda^A} \geq \frac{|\eta|}{1 - \beta^B} \) and
\[
\frac{\partial E^B_+[x^B_\eta]}{\partial \eta} = \frac{1}{1 - \beta^B} > 0. \tag{38}
\]

The redistributive policies of the parties have opposite characteristics. While the party with the lowest budget focuses on its partisan voters, whose vote are more easily obtained due to the favorable ideology bias, the party with the highest budget transfers more, in expectation, to the voters who are ideologically biased toward its opponent.

In the former case, a machine policy outcome emerges, and Party A reduces the expected transfer of voters ideologically biased toward the opponent by promising them a nil transfer with strictly positive probability. This allows it to reduce the fixed cost required to contest each of them, which is the amount \( \eta \).

In the latter case instead, Party B contests all the votes and in particular, all the votes of the people who are biased toward the opponent. This implies that it has to fully pay the aggregate fixed cost required to contest these votes, which amounts to \( \int_0^1 \phi(\eta) \eta \cdot d\eta \).

Party B reduces expected transfers to its partisan voters by promising them a nil transfer with strictly positive probability, and counting on the fact that for the same nil transfer received by both parties, these voters choose to vote for it, due to their ideological biases.

This result is in stark contrast with that obtained in the symmetric parties setting, where both parties pursue a swing voter strategy in their redistributive policies. In this case, on the contrary, there is a favored minority which receives the highest expected transfers from both parties, i.e. the share of the electorate which is ideologically biased toward Party A.

Consider now the effects of variations in the dispersion of the ideology biases in the electorate, measured in the present setting by the variance of the distribution of \( \eta \).
Corollary 6 An increase in the dispersion of ideology causes:

- an increase in the expected transfers promised by Party A to its own partisan voters and to the voters who have an ideology bias in absolute value, smaller than:

\[
|\eta| \leq \frac{2}{\lambda A} \cdot \sqrt{\frac{1}{\lambda A} \sqrt{1 + \frac{1}{2} \cdot \frac{\text{var}[\eta]}{(1-\beta B)^2} - 1}} - 1
\]  

(39)

and a decrease in the expected transfers of voters whose ideology bias exceeds this threshold; the increase in the expected transfers of partisan voters is always larger than that experienced by voters who are ideologically biased toward Party B.

- a decrease in the expected transfer promised by Party B to all voters; this decrease is larger for the voters who are partisan of Party A.

**Proof.** Since the size of party budgets is unaffected by variations in the dispersion of ideology, we can focus on the budget shares offered by the parties to the voters. Consider initially \( E^A[x^A_n] \), and note that

\[
\frac{\partial E^A_{-}[x^A_n]}{\partial \text{var}[\eta]} = \frac{1}{2} \cdot \frac{\partial}{\partial \text{var}[\eta]} \left[ 1 + \frac{1}{2} \cdot \frac{\text{var}[\eta]}{(1-\beta B)^2} \right]^{-\frac{3}{2}} \left[ 1 + \left( \frac{\lambda A \cdot \eta}{2} \right)^2 \right] +
\]

\[
+ \frac{1}{8} \cdot \frac{1}{\lambda A} \left[ 1 + \frac{1}{2} \cdot \frac{\text{var}[\eta]}{(1-\beta B)^2} \right]^{-\frac{3}{2}} \cdot \frac{1}{(1-\beta B)^2} \left[ 1 - \left( \frac{\lambda A \cdot \eta}{2} \right)^2 \right]
\]  

(40)

Note now that it is

\[
\frac{\partial}{\partial \text{var}[\eta]} = -\frac{1}{4} \left( \frac{1}{1-\beta B} \right)^2 \left[ 1 + \frac{1}{2} \cdot \frac{\text{var}[\eta]}{(1-\beta B)^2} \right]^{-\frac{1}{2}} < 0
\]  

(41)

and substitute (41) into (40) to obtain using simple algebra

\[
\frac{\partial E^A_{-}[x^A_n]}{\partial \text{var}[\eta]} = -\frac{1}{8} \left( \frac{1}{1-\beta B} \right)^2 \left[ 1 + \frac{1}{2} \cdot \frac{\text{var}[\eta]}{(1-\beta B)^2} \right]^{-1} \left[ 1 + \left( \frac{\lambda A \cdot \eta}{2} \right)^2 \right] +
\]

\[
+ \frac{1}{8} \cdot \frac{1}{\lambda A} \left[ 1 + \frac{1}{2} \cdot \frac{\text{var}[\eta]}{(1-\beta B)^2} \right]^{-\frac{3}{2}} \cdot \frac{1}{(1-\beta B)^2} \left[ 1 - \left( \frac{\lambda A \cdot \eta}{2} \right)^2 \right]
\]  

(42)

or reordering the terms
\[ \frac{\partial E^A[x^A]}{\partial \text{var}[\eta]} = \frac{1}{8} \left( \frac{1}{1 - \beta B} \right)^2 \left[ 1 + \frac{1}{2} \cdot \frac{\text{var}[\eta]}{(1 - \beta B)^2} \right]^{-1} \cdot \left( \frac{1}{\lambda A} \left[ 1 + \frac{1}{2} \cdot \frac{\text{var}[\eta]}{(1 - \beta B)^2} \right]^{\frac{1}{2}} - 1 \right) + \frac{1}{8} \left( \frac{1}{1 - \beta B} \right)^2 \left[ 1 + \frac{1}{2} \cdot \frac{\text{var}[\eta]}{(1 - \beta B)^2} \right]^{-1} \cdot \left( \frac{\lambda A \cdot \eta}{2} \right)^2 \left( 1 + \frac{1}{\lambda A} \left[ 1 + \frac{1}{2} \cdot \frac{\text{var}[\eta]}{(1 - \beta B)^2} \right]^{-\frac{1}{2}} \right). \]  

(43)

Note now that
\[ \frac{1}{\lambda A} \left[ 1 + \frac{1}{2} \cdot \frac{\text{var}[\eta]}{(1 - \beta B)^2} \right]^{-\frac{1}{2}} - 1 > 0 \]  

(44)
since by substituting the value of \( \frac{1}{\lambda A} \) in the previous inequality, this simplifies to \( \left[ 1 + \frac{1}{2} \cdot \frac{\text{var}[\eta]}{(1 - \beta B)^2} \right]^{-\frac{1}{2}} > 0 \). Hence it is always possible to find an \( \eta \) sufficiently small to have \( \frac{\partial E^A[x^A]}{\partial \text{var}[\eta]} > 0 \). In particular, by solving \( \eta \) in (43) we can characterize the threshold value for \( \eta \) which satisfies this inequality
\[ \frac{1}{\lambda A} \left[ 1 + \frac{1}{2} \cdot \frac{\text{var}[\eta]}{(1 - \beta B)^2} \right]^{-\frac{1}{2}} - 1 > 0 \]  

(45)
or simplifying and reordering the terms
\[ \frac{2}{\lambda A} \sqrt{\frac{\left[ 1 + \frac{1}{2} \cdot \frac{\text{var}[\eta]}{(1 - \beta B)^2} \right]^{-\frac{1}{2}} - 1}{\left[ 1 + \frac{1}{2} \cdot \frac{\text{var}[\eta]}{(1 - \beta B)^2} \right]^{-\frac{1}{2}}}} > |\eta|. \]  

(46)

Consider now \( E^A_+ [x^A_\eta] \)
\[ \frac{\partial E^A_+ [x^A_\eta]}{\partial \text{var}[\eta]} = \frac{1}{8} \left( \frac{1}{1 - \beta B} \right)^2 \left[ 1 + \frac{1}{2} \cdot \frac{\text{var}[\eta]}{(1 - \beta B)^2} \right]^{-1} \cdot \left( 1 + \frac{1}{\lambda A} \left[ 1 + \frac{1}{2} \cdot \frac{\text{var}[\eta]}{(1 - \beta B)^2} \right]^{-\frac{1}{2}} \right) > 0 \]
and note that \( \frac{\partial E_A^A[\sigma_A]}{\partial \var[\eta]} > \frac{\partial E_B^A[\sigma_A]}{\partial \var[\eta]} \).

Consider now Party \( B \)'s redistributive policies, and analyze expected transfers to partisan voters

\[
\frac{\partial E_B^B[x_A]}{\partial \var[\eta]} = \frac{1}{2} \cdot \frac{\partial \frac{1}{\var[x_A]}}{\partial \var[\eta]} \left[ 1 - \frac{1}{2} \left( \frac{\lambda_A \cdot \eta}{1 - \beta_B} \right)^2 \right] < 0 \tag{47}
\]

This inequality follows from the fact that \( \frac{\partial \frac{1}{\var[x_A]}}{\partial \var[\eta]} < 0 \) and

\[
1 - \frac{1}{2} \left( \frac{\lambda_A \cdot \eta}{1 - \beta_B} \right)^2 > 0
\]

hold since

\[
\frac{1}{\lambda_A} > \frac{\eta}{1 - \beta_B} \cdot \frac{1}{\sqrt{2}}
\]

for \( \forall \eta \).

Consider lastly that it is

\[
\frac{\partial E_B^B[x_A]}{\partial \var[\eta]} = \frac{1}{2} \cdot \frac{\partial \frac{1}{\var[x_A]}}{\partial \var[\eta]} < 0.
\]

This further implies that \( |\frac{\partial E_B^B[x_A]}{\partial \var[\eta]}| > |\frac{\partial E_B^B[x_A]}{\partial \var[\eta]}| \forall \eta \). ■

Variations in the degree of dispersion of ideology produce opposite effects on the redistributive policies of the parties. If in fact, an increase in \( \var[\eta] \) causes Party \( A \) to increase the expected transfers to its own partisan voters and to the voters who are less ideologically biased toward the opponent (while a decrease occurs in expected transfers to voters with the largest ideology bias for the opponent party), the reverse happens when Party \( B \)'s redistributive policies are considered. Party \( B \) reduces expected transfers to all voters and specially reduces the expected transfers to the partisan voters of Party \( A \).

This happens because an increase in ideology dispersion increases the weight in the electorate of voters with large ideology bias. As a consequence, for reasons similar to those described in the previous section for the symmetric party setting, Party \( A \) saves resources. The core supporters of the opponent party (with large ideology bias), in fact, receive the lowest expected transfers while all its partisan voters receive the same expected transfers. This implies that less resources are targeted to the former group of voters allowing to increase the expected transfers targeted to most of the rest of the electorate and specially to the core supporters of Party \( A \).

Given the strategy adopted by Party \( B \), an increase in ideology dispersion produces instead, opposite effects because it increases the weight
in the electorate of voters who receive high expected transfers, i.e. voters with large ideology bias toward Party A. This causes an increase in the need of resources by Party B. This effect is not balanced by the symmetric increase in the weight in the electorate of voters who have large ideology bias toward Party B, because Party B’s core supporters receive expected transfers which are lower than those received by Party A’s partisan voters. This fact requires Party B to reduce the expected transfers to all voters.

Consider now the effects of changes in the size of the budgets available to the parties, and note that in a context with additive ideology biases, it is the case that, differently from Sahuguet and Persico (2006), also the absolute size of party budgets is relevant and not only their relative size, i.e. the ratio $\rho$. This is the case because the absolute size of a party budget defines the share of it which is absorbed by the entry cost paid to contest the votes of people who are ideologically biased toward the opponent.

**Corollary 7** An increase in $\beta^A$ only affects the redistributive policies of Party A, and causes a reduction in the expected transfers of the voters which is proportional to the size of the expected budget share promised them by Party A.

**Proof.** Straightforwardly follows from Theorem (4) noting that $\frac{\partial \frac{1}{\beta^A}}{\partial \beta^A} = 0$.

An increase in the dead-weight loss associated to the redistributive policies of Party A causes a decrease in the expected transfers proportional to the budget share offered by Party A. These, as stated in Corollary (5), are increasing in the size of the ideology bias implying that the machine politics nature of its redistributive policies is blurred. Consider now the opposite case where Party B’s budget decreases due to an increase in the dead-weight loss associated with its redistributive policies.

**Corollary 8** An increase in $\beta^B$ affects both the redistributive policies of Party A and of Party B.

In particular, it causes an increase in the expected transfers which Party A targets to its partisan voters and to the voters whose ideology bias toward the opponent is smaller than the threshold

$$|\eta| \leq \frac{2}{\lambda^A} \cdot \frac{1}{\sqrt{\frac{1}{\lambda^A} \sqrt{1 + \frac{1}{2} \cdot \frac{\text{var}[\eta]}{(1-\beta^B)^2}} - 1}} - \frac{1}{\sqrt{\frac{1}{\lambda^A} \sqrt{1 + \frac{1}{2} \cdot \frac{\text{var}[\eta]}{(1-\beta^B)^2}} + 1}}. \quad (48)$$
In any case, the increase in the expected transfer is always larger for partisan voters. Voters whose ideology bias exceeds this threshold experience a reduction in the expected transfer promised by Party A.

With regard to Party B’s redistributive policies, an increase in $\beta^B$ causes a reduction in expected transfers to all voters. This reduction is larger in the case of voters who are ideologically biased toward Party A than in the case of Party B partisan voters.

**Proof.** Since the size of Party A’s budget is unaffected by variations in $\beta^B$ we can focus on the budget shares offered by Party A to the voters. Consider initially $E^A_{x^A_\eta}$ and note that

$$
\frac{\partial E^A_{x^A_\eta}}{\partial \beta^B} = \frac{1}{2} \cdot \frac{\partial \frac{1}{\lambda^B}}{\partial \beta^B} \left[ 1 + \frac{1}{2} \cdot \frac{\text{var}[\eta]}{(1 - \beta^B)^2} \right]^{-\frac{1}{2}} \left( 1 + \left( \frac{\lambda^A \cdot \eta}{2} \right)^2 \right) +
$$

$$
\frac{1}{4} \cdot \frac{1}{\lambda^A} \left[ 1 + \frac{1}{2} \cdot \frac{\text{var}[\eta]}{(1 - \beta^B)^2} \right]^{-\frac{3}{2}} \cdot \left( \frac{1}{1 - \beta^B} \right)^3 (\text{var}[\eta]) \left[ 1 - \left( \frac{\lambda^A \cdot \eta}{2} \right)^2 \right].
$$

(49)

Note now that it is

$$
\frac{\partial \frac{1}{\lambda^B}}{\partial \beta^B} = -\frac{1}{2} \left[ 1 + \frac{1}{2} \cdot \frac{\text{var}[\eta]}{(1 - \beta^B)^2} \right]^{-\frac{1}{2}} \frac{\text{var}[\eta]}{(1 - \beta)^3}
$$

(50)

and substitute (50) into (49) to obtain using simple algebra

$$
\frac{\partial E^A_{x^A_\eta}}{\partial \beta^B} = \frac{1}{4} \cdot \frac{\text{var}[\eta]}{(1 - \beta^B)^3} \left[ 1 + \frac{1}{2} \cdot \frac{\text{var}[\eta]}{(1 - \beta^B)^2} \right]^{-1}
$$

$$
\cdot \left( \frac{1}{\lambda^A} \left[ 1 + \frac{1}{2} \cdot \frac{\text{var}[\eta]}{(1 - \beta^B)^2} \right]^{-\frac{1}{2}} - 1 \right) +
$$

$$
-\frac{1}{4} \frac{\text{var}[\eta]}{(1 - \beta^B)^3} \left[ 1 + \frac{1}{2} \cdot \frac{\text{var}[\eta]}{(1 - \beta^B)^2} \right]^{-1} \left( \frac{\lambda^A \cdot \eta}{2} \right)^2
$$

$$
\cdot \left( 1 + \frac{1}{\lambda^A} \left[ 1 + \frac{1}{2} \cdot \frac{\text{var}[\eta]}{(1 - \beta^B)^2} \right]^{-\frac{1}{2}} \right).
$$

(51)

Since, as shown before the term $-1 + \frac{1}{\lambda^A} \left[ 1 + \frac{1}{2} \cdot \frac{\text{var}[\eta]}{(1 - \beta^B)^2} \right]^{-\frac{1}{2}}$ is positive, it is always possible to find an $|\eta|$ sufficiently small to have that $\frac{\partial E^A_{x^A_\eta}}{\partial \beta^B} > 0$. 

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holds. In particular, by solving $\eta$ in (51) it is possible to characterize the threshold value for $\eta$ for which this inequality holds:

$$
\frac{2}{\lambda^A} \sqrt{\frac{1}{\lambda^A} \left[ 1 + \frac{1}{2} \cdot \frac{\text{var}[\eta]}{(1 - \beta^B)^2} \right]^{-\frac{1}{2}} - 1} > |\eta|.
$$

(52)

Consider now $E^+_B[x^B_\eta]$ and note that

$$
\frac{\partial E^+_B[x^B_\eta]}{\partial \beta^B} = \frac{1}{4} \cdot \frac{\text{var}[\eta]}{(1 - \beta^B)^3} \left[ 1 + \frac{1}{2} \cdot \frac{\text{var}[\eta]}{(1 - \beta^B)^2} \right]^{-1} \cdot \\
\cdot \left( -1 + \frac{1}{\lambda^A} \left[ 1 + \frac{1}{2} \cdot \frac{\text{var}[\eta]}{(1 - \beta^B)^2} \right]^{-\frac{1}{2}} \right) > 0
$$

(53)

hence implying that $\frac{\partial E^+_B[x^B_\eta]}{\partial \beta^B} > \frac{\partial E^+_A[x^A_\eta]}{\partial \beta^A} \forall \eta$.

Analyze now the expected transfers promised by Party $B$. In this case, it is not enough to focus on the share of the budget promised by Party $B$ to the voters, but we need to consider

$$
\frac{\partial E^+_B[x^B_\eta](1 - \beta^B)}{\partial \beta^B} = \frac{1}{2} \cdot \frac{\partial \lambda^A}{\partial \beta^B} \left[ 1 + \left( \frac{\lambda^A \cdot \eta}{1 - \beta^B} \right) \right] (1 - \beta^B) + \\
+ \frac{1}{\lambda^A}(2) \left( \frac{\lambda^A \cdot \eta}{1 - \beta^B} \right) \frac{(-1)(\lambda^A)^2 \frac{\partial \lambda^A}{\partial \beta^B} \cdot \eta(1 - \beta^B) + \lambda^A \cdot \eta}{(1 - \beta^B)^2} (1 - \beta^B) + \\
+ (-1) \frac{\eta}{(1 - \beta^B)^2} (-1)(1 - \beta^B) - E^+_B[x^B_\eta]
$$

Substitute $E^+_B[x^B_\eta]$ in the previous equation to obtain, using simple algebra

$$
\frac{\partial E^-_B[x^B_\eta](1 - \beta^B)}{\partial \beta^B} = \frac{1}{2} \left[ \frac{\partial \lambda^A}{\partial \beta^B} (1 - \beta^B) - \frac{1}{\lambda^A} \right] \cdot \left[ 1 - \left( \frac{\lambda^A \cdot \eta}{1 - \beta^B} \right) \right] < 0
$$

(65)

since by (50) it is $\frac{\partial \lambda^A}{\partial \beta^B} < 0$.

Consider now

$$
\frac{\partial E^+_B[x^B_\eta](1 - \beta^B)}{\partial \beta^B} = \frac{1}{2} \cdot \frac{\partial \lambda^A}{\partial \beta^B} (1 - \beta^B) + \\
+ (-1) \frac{\eta}{(1 - \beta^B)^2} (-1)(1 - \beta^B) - E^+_B[x^B_\eta]
$$

(56)
or substituting $E^B_\eta[x^B_\eta]$ and simplifying and reordering the terms

$$
\frac{\partial E^B_+[x^B_\eta](1 - \beta^B)}{\partial \beta^B} = \frac{1}{2} \cdot \frac{\partial}{\partial \beta^B} \frac{1}{\lambda^B} (1 - \beta^B) + \frac{1}{2} \cdot \frac{1}{\lambda^A} < 0. \quad (57)
$$

It is easy to see that $\left| \frac{\partial E^B_+[x^B_\eta](1 - \beta^B)}{\partial \beta^B} \right| > \left| \frac{\partial E^B_+[x^B_\eta](1 - \beta^B)}{\partial \beta^A} \right| \forall \eta$. ■

A decrease in the size of Party B’s budget, due to an increase in the dead-weight loss associated to its redistributive policies, affects the expected transfers promised to the electorate by its opponent. In particular, Party A increases the expected transfer to its partisan voters while increasing to a lesser extent, the expected transfer of voters who are ideologically biased toward Party B. It may even decrease these transfers when their ideological bias is sufficiently high. Hence, an increase in $\beta^B$ emphasizes the machine politics outcome of Party A’s redistributive policies.

A similar dynamics is observed also in Party B’s redistributive policies since the reduction in the expected transfers of voters who are partisan of Party B is always smaller than that occurring in the expected transfer of voters who have an ideological bias for Party A. The swing voter outcome of these policies thus is blurred by an increase in $\beta^B$.

This effects are similar to those observed in the case of an increase in the dispersion of ideology. The reasons of this similarity are easily understood. An increase in the dead-weight loss in Party B’s redistributive policies increases the need of resources by Party B and weakens its strategic position vis-a-vis Party A. These are the same effects produced by an increase in ideology dispersion which increases the weight of voters who are ”‘costly’” to contest for Party B and ”‘cheap’” for Party A.

### 6 Final remarks

The present paper analyzes the equilibrium of the redistribution game which characterizes the electoral competition between two parties in a proportional system, and focuses on the effects on the equilibrium of the game of the interactions between party valence, measured by the dead-weight loss in party redistributive policies, and voter ideology. In particular, the present analysis considers two distinct settings which share the common feature that ideology is symmetrically distributed in the electorate. In the first setting the competing parties are symmetric also with respect to the budget available for redistribution, while in the second setting we have a favored party whose budget is larger than that of its opponent.

The setting with symmetric parties is characterized by an equilibrium where party redistributive policies follow a swing voter strategy,
implying that the voters who have the lowest ideology biases receive, in expectation, the highest transfers form the parties. Moreover, for given absolute value of the ideology bias, core supporters of a party receive an expected transfer which is larger than that targeted to voter who are partisan of the opponent party. In this context, an increase in the dispersion of ideology causes an increase in party expected transfers to all voters with the highest increases involving the voters who are ideologically biased toward the opponent party.

These results change quite substantially when an asymmetry in the size of party budgets available for redistribution is introduced. This is specially true if the equilibrium strategy of the party with the highest budget, i.e. Party $B$, is considered. In particular, we observe that its redistributive policies reserve the highest expected transfers to the voters who are partisan of the opponent party and whose ideology bias is the highest, while Party $B$’s core supporters receive less the higher is their ideology bias. Moreover an increase in ideology dispersion does not cause a generalized increase in expected transfers but quite the opposite. Party $B$, in fact, reduces expected transfers to all voters and particularly those targeted to the partisan voters of Party $A$.

A greater similarity with the results of the symmetric party setting emerge instead when we consider the equilibrium strategy adopted by the party with the lowest budget, Party $A$. Its expected transfers, in facts, are weakly increasing in the size of the ideology bias and swing voters receive the highest expected transfer. This is also the same expected transfer received by Party $A$’s partisan voters irrespectively of the size of their ideological bias. The analogy also concerns the results obtained when an increase in ideology dispersion occurs. In facts, Party $A$’s redistributive policies change, and involve an increase of the expected transfers targeted to most of the voters in the electorate (except those with very high ideology biases toward the opponent), analogously to what happens in the symmetric party setting.

What justifies these differences concerning the strategy adopted by the party with the largest budget, is the fact that Party $B$, having a larger amount of resources available for redistribution than its opponent, contests all the votes in the electorate by offering a strictly positive transfers to every voter, also to those who are partisan of Party $A$. This never happens when parties are symmetric, since in the groups where voters are biased toward the opponent, the redistributive policies of the parties always involve a nil transfer with strictly positive probability, which is equivalent to say that a share of these voters is in fact not contested. The fact that the party with the largest budget competes for all the votes in the electorate implies that it is forced to fully pay
the "entry" cost represented by the ideology biases of all the people who are partisan of the opponent party. Since this cost increases with the size of the ideology bias, these voters are also those receiving the highest expected transfers. This further implies that an increase in their weight in the electorate, following an increase in ideology dispersion, also increases the entry cost paid by Party B, which in turn is forced to decrease the expected transfers to all voters.

Our analysis further considers an element which specific to the asymmetric party setting, i.e. the variation in party budget available for redistribution following a change in the dead-weight loss associated to party policies. In this context, a decrease in the budget of the favored party has the same effects of an increase in ideology dispersion. The reason is that in fact, it weakens the position of Party B by increasing its need for resources (just like an increase in the entry cost required to contest the partisan of the opponent party), and strengthen the position of the party with the lowest budget. A decrease in Party A’s budget available for redistribution, lastly, has little impact on the strategy of the parties, in the sense that it does not change the equilibrium distribution of the share of the budget which each party offers to each voter. This implies that it has no impact also on the expected transfers promised by Party B to the voters, and causes a reduction in the expected transfers of Party A which is proportional to the budget share promised to each voter.

7 Appendix

7.1 Connections with all-pay auctions: symmetric parties

Consider initially the connection with all-pay contests. Start from the maximization problem of Party i (i = A, B) which states that it obtains the votes of people who receive an offer which guarantees a level of utility higher than that guaranteed by the offer of the opponent. Note now that, in this context, Party i optimally distributes the whole public budget implying that the budget constraint is always binding This allows, using simple algebra, to rewrite Equation (1) as

\[
\max_{F_i(x_i)} \int_{-\bar{\eta}}^{\bar{\eta}} \phi(\eta) \lambda \int_0^\infty \left( \frac{1}{\lambda} F_j^i (x_i + \eta) - x_\eta^i \right) \cdot dF_j^i (x_\eta^i) \cdot d\eta + \lambda.
\]

This rewriting of the maximization problem shows that up to a linear transformation, it is equivalent to the problem of a risk-neutral player, Player i corresponding to Party i, who maximizes expected utility by competing with Player j, corresponding to Party j in a continuum of independent first-price all-pay auctions. Each auction is played with a
frequency equal to \( \phi(\eta) \) and awards the same prize whose appraisal by Party \( i \) is \( \frac{1}{\lambda} \), where \( \lambda \) is the optimal choice for the Lagrangian multiplier attached to the budget constraint in Party \( i \)'s maximization problem. Note that in each auction the players are asymmetric because a headstart of size \( |\eta| \) is awarded to one of them. In particular whenever \( \eta > 0 \) Player \( A \) is the favored player while the reverse occurs when \( \eta < 0 \). When \( \eta = 0 \) players are symmetric. In this context, the quantity, \( \frac{1}{\lambda} \pm \eta \) corresponds to what Siegel (2009) defines a player reach.

7.2 Proof of Theorem 1

Consider Party \( A \)'s redistributive policies, and analyze a generic voter with an ideology bias \( \eta \geq 0 \). Successive rounds of elimination of dominated actions restrict party offers to the interval \([0, \frac{1}{\lambda} - \eta]\) for Party \( A \) and to \([\eta, \frac{1}{\lambda}] \cup \{0\}\) for Party \( B \). Note indeed, that \( 0 < x^B_\eta \leq \eta \) and \( x^B_\eta > \frac{1}{\lambda} \) are dominated by \( x^B_\eta = 0 \). This allows Party \( A \) to obtain the vote of a voter with ideology bias \( \eta \) with certainty, by offering \( \frac{1}{\lambda} > x^A_\eta \geq \frac{1}{\lambda} - \eta \). However, \( x^A_\eta = \frac{1}{\lambda} - \eta \) dominates every greater offer, which reduces the resources available for redistribution without increasing the probability of obtaining this vote.

The standard argument by Hillman and Riley (1989) proves that the parties randomize continuously over the intervals characterized above. A strictly positive probability can only be attached to \( x^A_\eta = 0, x^B_\eta = \eta \) or to \( x^B_\eta = 0 \). However, it cannot be the case that \( F^A(0) > 0 \) holds, and Party \( B \) randomizes over a support which includes \( x^B_\eta = \eta \). Since ties are broken randomly, Party \( B \)'s probability of obtaining the vote of a member of group \( k \) decreases discontinuously at \( \eta \), and a profitable deviation exists, i.e. to shift probability to \( x^B_\eta > \eta \). A symmetrical argument excludes that \( F^B(\eta) > 0 \) holds, and that, at the same time, Party \( A \) randomizes over a support which includes 0. From these results it further follows that the probability of a tie is nil, since party strategies are continuous over \((0, \frac{1}{\lambda} - \eta]\) and \((\eta, \frac{1}{\lambda}]\).

Let now \( \bar{x}^i_\eta \) be the upper bound for the support of Party \( i \)'s distribution. In equilibrium, the equality \( \bar{x}^A_\eta = \bar{x}^B_\eta - \eta \) holds. Suppose instead that \( \bar{x}^A_\eta > \bar{x}^B_\eta - \eta \). When \( x^A_\eta = \bar{x}^A_\eta \), Party \( A \) obtains the vote of a member of group \( k \) with probability 1; every offer such that \( \bar{x}^B_\eta - \eta < x^A_\eta < \bar{x}^A_\eta \) has the same probability of obtaining this vote but increases the resources available for redistributive policies and dominates \( \bar{x}^A_\eta \). A symmetric argument applied to Party \( B \) excludes that \( \bar{x}^A_\eta < \bar{x}^B_\eta - \eta_k \) holds.

Let \( x^i_\lambda \) denote the lower bound of Party \( i \)'s support. The equality

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9The equilibrium strategies of the all-pay auction setting, which are equivalent to party redistributive policies are characterized in Konrad (2002) and Magnani (2010).

10On this see also Baye et al. (1996), Che and Gale (1998) and Ellingsen (1991).
$x^A = x^B - \eta$ must hold in equilibrium. Suppose that, instead, $x^B_\eta < x^A_\eta - \eta_k$. Any offer such that $x^A_\eta \leq x^A < x^B_\eta - \eta$ gives Party A a nil probability of obtaining the vote of a voter with ideology bias $\eta$, and is dominated by $x^A = 0$. Assume now that $x^B_\eta > x^A_\eta - \eta$; a symmetric argument excludes that this inequality is verified.

Note further that $x^A_\eta = 0$ and $x^B_\eta = \eta_k$ hold. Indeed, if $x^A_\eta > 0$, Party A can increase its resources available for redistributive policies by shifting down the lower bound of the distribution. This does not affect the probability of obtaining the vote of a voter with ideology bias since $x^A_\eta = x^B_\eta$ and $F^B_\eta (x^B_\eta) = 0$ hold.

In equilibrium, the parties obtain the same expected payoff from each pure strategy over which they randomize. So in the case of Party A, the difference between the expected utility from obtaining the vote of a voter with ideology bias and its offer must not vary, i.e.

$$\frac{1}{\lambda} F^A_\eta (x^A + \eta) - x^A = p^A_\eta$$

holds, where $p^A_\eta$ is a specific constant which depends on $\eta$; the same argument applies also to Party B, and $p^B_\eta$ is analogously defined. Note now that, since Party B never offers more than $\frac{1}{\lambda}$, and $\frac{1}{\lambda} \leq \frac{1}{\lambda} + \eta$ holds, Party A can always obtain $p^A_\eta > 0$ by offering $x^A_\eta \geq \frac{1}{\lambda}$. This implies that in equilibrium $p^A_\eta > 0$, and when $x^A_\eta = 0$ this requires $F^B_\eta (\eta) > 0$. As a consequence, $p^B_\eta = 0$ must hold. Suppose that this is not the case; $p^B_\eta > 0$ requires that if $x^B_\eta = \eta$, $F^A_\eta (0) > 0$ holds. However, the inequality $F^B_\eta (\eta) > 0$ must be verified as well, implying that the atom in Party B’s distribution is placed at 0 as proven above. But $x^B_\eta = 0$ always gives a nil probability of obtaining the vote of a voter with ideology bias $\eta$, and this contradicts $p^B_\eta > 0$.

Since $p^B_\eta = 0$, in equilibrium $\bar{x}^A_\eta = \frac{1}{\lambda} - \eta$ and $\bar{x}^B_\eta = \bar{x}^A + \eta = \frac{1}{\lambda}$ hold. Suppose in fact that $\bar{x}^B_\eta < \frac{1}{\lambda}$, and note that this implies $\bar{x}^A_\eta < \frac{1}{\lambda} - \eta$. Party B then has the profitable deviation to offer slightly more than $\bar{x}^B_\eta$ to get $p^B_\eta > 0$.

The equilibrium conditions may now be defined. The following equality must be verified:

$$\frac{1}{\lambda} F^A_\eta (x^B_\eta - \eta) - x^B_\eta = 0$$

so that $F^A_\eta (x^B_\eta - \eta) = \frac{x^B_\eta - \eta}{\lambda} + \frac{2}{\lambda}$ holds or, exploiting the equality $x^A_\eta = x^B_\eta - \eta$,

$$F^A_\eta (x^A_\eta) = \lambda (x^A_\eta + \eta).$$
Since no negative offers are allowed, Party A’s equilibrium distribution is a uniform over the interval \((0, \frac{1}{\lambda} - \eta]\) with an atom in 0 amounting to \(\eta\).

An offer \(x^A_\eta = \frac{1}{\lambda} - \eta\) implies that \(p^A_\eta = \eta\) holds so that in equilibrium the condition:

\[
\frac{1}{\lambda} F^B_\eta(x^A_\eta + \eta) - x^A_\eta = \eta
\]

is verified or

\[
F^B_\eta(x^B_\eta) = \lambda \cdot x^B_\eta.
\]

Given that \(x^B_\eta = \eta\), the previous distribution also has an atom of probability. Since \(F^A_\eta(0) > 0\) holds, the atom must be placed in 0. Party B thus offers 0 with probability \(\lambda \cdot \eta \geq 0\), and randomizes according to the cumulative distribution function \(\lambda \cdot x^B_\eta\) over the support \((\eta, \frac{1}{\lambda}]\), with complementary probability.

Uniqueness of these strategies for a given level of \(\frac{1}{\lambda}\) follows from the standard argument proposed by Baye et al. (1996) in the all-pay setting. Uniqueness of the equilibrium follows then from the argument that there exists a unique value of \(\frac{1}{\lambda}\) which satisfies the budget constraint for the equilibrium strategies defined above. This value of \(\frac{1}{\lambda}\) is obtained by the following equation:

\[
1 = \int_{-\tilde{\eta}}^{0} \phi(\eta) \left( \int_{-\tilde{\eta}}^{\frac{1}{\lambda} - \tilde{\eta}} \lambda x^i_\eta \, dF^i_\eta(x^i_\eta) \right) \, d\eta + \int_{0}^{\tilde{\eta}} \phi(\eta) \left( \int_{0}^{\frac{1}{\lambda} - \tilde{\eta}} \lambda x^i_\eta \, dF^i_\eta(x^i_\eta) \right) \, d\eta
\]

or, using simple algebra:

\[
1 = \int_{-\tilde{\eta}}^{0} \phi(\eta) \frac{1}{2\lambda} \, d\eta + \int_{0}^{\tilde{\eta}} \phi(\eta) \frac{\lambda \eta^2}{2} \, d\eta - \int_{-\tilde{\eta}}^{\tilde{\eta}} \phi(\eta) \frac{\lambda \eta^2}{2} \, d\eta + \int_{0}^{\tilde{\eta}} \phi(\eta) \, d\eta.
\]

By the symmetry if the distribution

\[
\int_{-\tilde{\eta}}^{0} \phi(\eta) \frac{\lambda \eta^2}{2} \, d\eta = \int_{0}^{\tilde{\eta}} \phi(\eta) \frac{\lambda \eta^2}{2} \, d\eta
\]

implying that
\[ 1 = \frac{1}{2\lambda} d\eta - \int_{0}^{\bar{\eta}} \phi(\eta) \eta d\eta \]  
(61)

and thus
\[ \frac{1}{\lambda} = 2 + 2 \cdot \int_{0}^{\bar{\eta}} \phi(\eta) \eta d\eta \]  
(62)

Note that since \( \bar{\eta} < 2 \) in facts \( \frac{1}{\lambda} \geq \eta \) always holds.

By the symmetry of party strategies it further follows that \( S^A = S^B = \frac{1}{2} \).

7.3 Connections with all-pay auctions: asymmetric parties

Consider initially the connection with all-pay contests when the two parties have different budgets, and start from the maximization problem of Party \( i \) (\( i = A, B \)) reported in Equation (15). Define now \( y^A_\eta = \rho \cdot x^A_\eta \)
and \( \tilde{F}_\eta^A(\frac{y^A_\eta}{\rho}) \) and note that also in this context, the parties optimally distribute the whole public budget implying that the budget constraints are always binding. This allows, using simple algebra, to rewrite the maximization problem of Party \( A \) as

\[
Max_{\tilde{F}_\eta^A(y^A_\eta)} \int_{-\bar{\eta}}^{\bar{\eta}} \phi(\eta) \int_{0}^{\infty} \tilde{F}_\eta^B \left( y^A_\eta + \frac{\eta}{1 - \beta_B} \right) \cdot d\tilde{F}_\eta^A(y^A_\eta) \cdot d\eta + \\
+ \lambda_A \left[ \rho - \int_{-\bar{\eta}}^{\bar{\eta}} \phi(\eta) \left( \int_{0}^{\infty} y^A_\eta \cdot d\tilde{F}_\eta^A(y^A_\eta) \right) \cdot d\eta \right]
\]

or, reordering the terms

\[
Max_{\tilde{F}_\eta^A(y^A_\eta)} \int_{-\bar{\eta}}^{\bar{\eta}} \phi(\eta) \lambda_A \int_{0}^{\infty} \left( \frac{1}{\lambda_A} \tilde{F}_\eta^B \left( y^A_\eta + \frac{\eta}{1 - \beta_B} \right) - y^A_\eta \right) d\tilde{F}_\eta^A(y^A_\eta) \cdot d\eta + \lambda_A \rho.
\]

This rewriting of the maximization problem shows that up to a linear transformation, it is equivalent to the problem of a risk-neutral player, Player \( A \) corresponding to Party \( A \), who maximizes expected utility by competing with Player \( B \), corresponding to Party \( B \) in a continuum of independent first-price all-pay auctions.

Following the same steps it is also possible to rewrite Party \( B \)’s maximization problem as follows

\[
Max_{\tilde{F}_\eta^B(y^B_\eta)} \int_{-\bar{\eta}}^{\bar{\eta}} \phi(\eta) \lambda_A \int_{0}^{\infty} \left( \frac{1}{\lambda_A} \tilde{F}_\eta^A \left( x^B_\eta + \frac{\eta}{1 - \beta_B} \right) - x^B_\eta \right) d\tilde{F}_\eta^B(\eta) \cdot d\eta + \lambda_B.
\]
Each auction is played with a frequency equal to \( \phi(\eta) \) and awards the same prize whose appraisals are respectively \( \frac{1}{\lambda_A} \) for Player A and \( \frac{1}{\lambda_B} \) for Player B, where \( \lambda_i \) is the optimal choice for the Lagrangian multiplier attached to the budget constraint in Party \( i \)'s maximization problem. Note that in each auction the players are asymmetric because an head-start of size \( |\eta| \) is awarded to one of them. In particular whenever \( \eta > 0 \) Player A is the favored player while the reverse occurs when \( \eta < 0 \). When \( \eta = 0 \) players no favored player exists. In this context, the quantity, \( \frac{1}{\lambda_i} \pm \eta \) corresponds in the definition by Siegel (2009) to Player \( i \)'s reach.

### 7.4 Proof of Theorem (4)

For what concerns the definition of the strategies of the parties, the proof is analogous to the proof of Theorem (1). Consider now the characterization of the values of the inverse of the Lagrangeans attached to the budget constraints of the parties. These values must satisfy the following system of equations

\[
\begin{align*}
1 &= \int_{-\eta}^{0} \phi(\eta) \left( \int_{\frac{1}{\lambda_A}}^{\frac{1}{\lambda_A} + \frac{1}{\rho}} \rho \cdot \lambda^B \cdot x^A \right) \cdot \,d\eta + \int_{0}^{\eta} \phi(\eta) \left( \int_{\frac{1}{\lambda_B}}^{\frac{1}{\lambda_B} + \frac{1}{\rho}} \lambda^B \cdot x^A \right) \cdot \,d\eta \\
1 &= \int_{-\eta}^{0} \phi(\eta) \left( \int_{\frac{1}{\lambda_A}}^{\frac{1}{\lambda_A} + \frac{1}{\rho}} \lambda^A \cdot x^B \right) \cdot \,d\eta + \int_{0}^{\eta} \phi(\eta) \left( \int_{\frac{1}{\lambda_A}}^{\frac{1}{\lambda_A} + \frac{1}{\rho}} \lambda^A \cdot x^B \right) \cdot \,d\eta
\end{align*}
\]

or

\[
\begin{align*}
2 \cdot \frac{1}{\lambda_A} &= \int_{-\eta}^{0} \phi(\eta) \left( \frac{1}{\lambda_A} \cdot \frac{1}{\rho} \right) \cdot \,d\eta - \int_{-\eta}^{0} \phi(\eta) \left( \frac{1}{\lambda_A} \cdot \frac{1}{\rho} \right) \cdot \,d\eta + \int_{0}^{\eta} \phi(\eta) \left( \frac{1}{\lambda_A} \cdot \frac{1}{\rho} \right) \cdot \,d\eta - \int_{0}^{\eta} \phi(\eta) \left( \frac{1}{\lambda_A} \cdot \frac{1}{\rho} \right) \cdot \,d\eta
\end{align*}
\]

Rewrite now the previous system as follows

\[
\begin{align*}
2 \cdot \frac{1}{\lambda_A} &= \int_{-\eta}^{\eta} \phi(\eta) \left( \frac{1}{\lambda_A} \cdot \frac{1}{\rho} \right) \cdot \,d\eta - \int_{-\eta}^{0} \phi(\eta) \left( \frac{n - E[\eta]}{1 - \beta^2} \right) \cdot \,d\eta \\
2 \cdot \frac{1}{\lambda_A} &= \int_{-\eta}^{\eta} \phi(\eta) \left( \frac{1}{\lambda_A} + \frac{n - E[\eta]}{1 - \beta^2} \right) \cdot \,d\eta - \int_{0}^{\eta} \phi(\eta) \left( \frac{n - E[\eta]}{1 - \beta^2} \right) \cdot \,d\eta
\end{align*}
\]

and note that, by the symmetry if the distribution of ideology it is

\[
\int_{0}^{\eta} \phi(\eta) \left( \frac{n - E[\eta]}{1 - \beta^2} \right) \cdot \,d\eta = \frac{1}{2} \cdot \text{var}[\eta]
\]

implying further that

\[
\begin{align*}
2 \cdot \frac{1}{\lambda_A} &= \int_{-\eta}^{\eta} \phi(\eta) \left( \frac{1}{\lambda_A} \cdot \frac{1}{\rho} \right) \cdot \,d\eta - \frac{1}{2} \cdot \frac{\text{var}[\eta]}{(1 - \beta^2)^2} \\
2 \cdot \frac{1}{\lambda_A} &= \int_{-\eta}^{\eta} \phi(\eta) \left( \frac{1}{\lambda_A} + \frac{n - E[\eta]}{1 - \beta^2} \right) \cdot \,d\eta - \frac{1}{2} \cdot \frac{\text{var}[\eta]}{(1 - \beta^2)^2}
\end{align*}
\]

or simplifying and reordering the terms

\[
\begin{align*}
2 \cdot \frac{1}{\lambda_A} &= \left( \frac{1}{\lambda_A} \cdot \frac{1}{\rho} \right) \cdot \,d\eta - \frac{1}{2} \cdot \frac{\text{var}[\eta]}{(1 - \beta^2)^2} \\
0 &= \left( \frac{1}{\lambda_A} \cdot \frac{1}{\rho} \right) \cdot \,d\eta + \frac{1}{2} \cdot \frac{\text{var}[\eta]}{(1 - \beta^2)^2}
\end{align*}
\]
Solve now $\frac{1}{x^a}$ in the second equation to obtain

\[
\begin{aligned}
\frac{2}{\rho} \cdot \frac{1}{x^a} &= \left( \frac{1}{x^a} \cdot \frac{1}{\rho} \right)^2 - \frac{1}{2} \cdot \frac{\text{var}[\eta]}{(1-\beta^B)^2} \\
\frac{1}{x^a} &= 1 + \sqrt{1 - \frac{1}{2} \cdot \frac{\text{var}[\eta]}{(1-\beta^B)^2}}
\end{aligned}
\]

The last step derives from the observation that for $\text{var}[\eta] = 0$ it must be the case that $\frac{1}{x^a} = 2$. This result is proven by Sahuguet and Persico (2006). Note now that the previous definition of $\frac{1}{x^a}$ requires that

\[
1 \geq \frac{1}{2} \cdot \frac{\text{var}[\eta]}{(1-\beta^B)^2}
\]

Since $\text{var}[\eta] \leq (\bar{\eta})^2$, a sufficient condition for the previous inequality to hold is

\[
(1 - \beta^B) \sqrt{2} \geq \bar{\eta} \quad (63)
\]

Substituting now $\frac{1}{x^a}$ in the previous system of equation allows to solve $\frac{1}{x^a}$ in the first equation

\[
\begin{aligned}
\frac{2}{\rho} \cdot \frac{1}{x^a} &= \left( \frac{1}{\rho} \right)^2 + \frac{1}{2} \cdot \frac{\text{var}[\eta]}{(1-\beta^B)^2} + \frac{1}{2} \cdot \var{\eta} \cdot \text{var}[\eta] \cdot \text{var}[\eta] \cdot \text{var}[\eta] \\
\frac{1}{x^a} &= 1 + \sqrt{1 - \frac{1}{2} \cdot \frac{\text{var}[\eta]}{(1-\beta^B)^2}}
\end{aligned}
\]

or simplifying and reordering the terms

\[
\begin{aligned}
\frac{1}{x^a} &= \frac{1}{x^a} \cdot \frac{1}{\rho} \sqrt{1 - \frac{1}{2} \cdot \frac{\text{var}[\eta]}{(1-\beta^B)^2}} \\
\frac{1}{x^a} &= 1 + \sqrt{1 - \frac{1}{2} \cdot \frac{\text{var}[\eta]}{(1-\beta^B)^2}}
\end{aligned}
\]

Consider now party expected vote shares and note that each party obtains the vote of half of the voters to whom it offers a positive transfer. This implies that Party $A$’s vote share is defined as follows:

\[
S^A = \frac{1}{2} \int_{-\tilde{\eta}}^{0} \phi(\eta) \left[ \frac{\lambda^B}{\lambda^A} - \rho \cdot \lambda^B \cdot \left( -\frac{\eta}{1 - \beta^A} \right) \right] d\eta + \frac{1}{2} \int_{0}^{\tilde{\eta}} \phi(\eta) \left( \frac{\lambda^B}{\lambda^A} \right) d\eta
\]

\[
S^A = \frac{1}{2} \int_{-\tilde{\eta}}^{0} \phi(\eta) \lambda^B \cdot \frac{\eta}{1 - \beta^B} d\eta + \frac{1}{2} \int_{-\tilde{\eta}}^{\tilde{\eta}} \phi(\eta) \left( \frac{\lambda^B}{\lambda^A} \right) d\eta
\]

\[
S^A = \frac{1}{2} \cdot \frac{\lambda^B}{1 - \beta^B} \int_{-\tilde{\eta}}^{0} \phi(\eta) \eta d\eta + \frac{1}{2} \cdot \frac{\lambda^B}{\lambda^A}
\]
References


