Altruism and Intergenerational State Education and Pensions†

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Abstract

This paper links altruism to intergenerational state human capital, physical capital and pensions in a three-overlapping-generations model. We find that the extent of altruism determines the regimes of suboptimality in human and physical capital accumulation. Regarding the policy to restore optimality, under a large extent of altruism and thus operative bequest motives, it optimal to subsidize both education and bequests financed by lump-sum taxes of the working age if the altruism factor is smaller than the social weight, but to tax both if otherwise. In this case, public education financing and public pensions do not need. By contrast, under a small extent of altruism and thus inoperative bequest motives, it is optimal to subsidize education and transfer the retirees a lump sum, financed by taxing the working age a lump sum. In this case, education subsidies and public pensions are necessary. Our quantitative exercise suggests that inoperative bequest motives are more in line with the data. Therefore, the rationale for public education financing and public pensions is strong.

Keywords: Human capital, Intergenerational transfers, Bequests, Pension, Education subsidy.
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1. Introduction

Parental altruism is an important factor affecting decisions on savings and human capital investment. This paper links the suboptimality of physical and human capital accumulation to altruism in a three-period overlapping generations (OLG) economy. Since Diamond (1965) it has been well-known that the accumulation of physical capital in equilibrium needs not be efficient. Recent literature has stressed the importance of human capital on economic growth.\(^1\) If human capital investment is considered, the efficiency issue becomes complicated. Both types of capital may under- or over-accumulate, or one type under-accumulates and the other type over-accumulates in equilibrium, as compared to the social optimum. Then, policy instruments such as education subsidies and pension benefits may be useful to correct allocation inefficiencies. Boldrin and Montes (2005), Docquier et al. (2007) and Bishnu (2013) have provided policy analysis in a normative approach.

Boldrin and Montes (2005) is a three-period OLG model, wherein agents borrow to pay education in order to accumulate human capital when young, work, consume and save when middle-aged, and retire and consume when old. These authors did not analyze the social planner’s problem. Instead, they assumed a missing credit market for human capital investment. They found that the long-run competitive equilibrium features under-accumulated human capital along with over-accumulated physical capital. They suggested the use of public education financing and public pensions to support the complete market allocation.

Later, Docquier et al. (2007) extended Boldrin and Montes (2005) to one with a perfect credit market for human capital investment and human capital externalities. They compared the equilibrium allocation with the social optimum, with the social planner maximizing the sum of utilities over generations discounted by a social weight. They found three accumulation regimes: under-accumulated human and physical capital if the social weight is high, over-accumulated human and physical capital if the social weight is low, and under-accumulated human capital and over-accumulated physical capital if the social weight is in between. Although the policy of public education financing and public pensions may reach optimality, they argued that the rationale for public pensions is weak when the social weight is high.

Finally, Bishnu (2013) extended Docquier et al. (2007) to an otherwise the same model except human capital externalities but with consumption externalities. He uncovered two accumulation regimes: both physical and human capital under-accumulate if the social weight is high, and both over-accumulate if otherwise. By restricting to the policy of intergenerational arrangements, Bishnu (2013) suggested implementing either public education financing or public pensions, but not both at the same time.\(^2\)

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\(^1\) On economic growth, see Lucas (1988) and Romer (1990) for theoretical literature and Levine and Renelt (1992) and Barro and Lee (1993) for empirical work.

\(^2\) There is a different strand, which adopts a positive approach to analyze the effects of education subsidies and
Families can redistribute income and extend credit among their members. Family exchange is a fundamental economic issue. Family exchange can be explained by altruistic motives. Altruistic behavior appears most obviously in kin relationships, especially in parenting. Parental altruism is an important factor influencing decisions on savings and human capital investment.\(^3\) Parental altruism involves bequests and education financing for children.

The literature of parental bequests dates back the early contribution by Barro (1974), who showed that parental bequests lead to the Ricardian equivalence of the debt neutrality. Later, Buiter (1979) and Carmichael (1982) found that the debt neutrality proposition hinges on whether or not bequest motives are operative. Weil (1987) formally determined necessary and sufficient conditions that bequest motives are operative or inoperative. Abel (1988) analyzed the effect of fiscal policy on the capital accumulation under operative and inoperative bequest motives.\(^4\) Regarding education financing for children, existing studies found that parents are willing to pay for children’s high school education and beyond (Steelman and Powell, 1991). Surveys carried out by the Sallie Mae Education Institute disclosed that financing children’s college education was among the parents’ most important investment they can make (Miller, 1997).\(^5\) Using Consumer Expenditure Survey over the period of 1972-2007, Kornrich and Furstenberg (2007) uncovered that about 50% of parental spending per child went to education. Moreover, not just parents in developed countries, parents in developing countries are willing to finance children’s education. Using 1985-1986 Peru living Standard Survey, Gertler and Glewwe (1990) revealed that parents in Peru, and even in rural areas, were willing to pay for children’s education.\(^6\)

The effectiveness and intergenerational transmission of the tax policy hinge on the extent of altruism. However, the above papers have ignored this consideration. This paper has two purposes. First, we study how the extent of altruism affects the suboptimality of physical and human capital accumulation. Then, we investigate whether the policy of public education financing and public pensions can help restore the optimality.

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3 See Barro (1974), Altonji et al. (1997), Nishiyama (2002) and Fuster et al. (2003), among others, for discussion concerning the role and the effects of family exchange and altruism.

4 Other than pure altruism used in Barro (1974), several alternative bequest motives have been proposed, including incomplete annuity markets (Abel, 1985), strategic bequest behavior (Bernheim et al., 1985), joy of giving (Abel, 1988) and warm-glow giving (Andreoni, 1990). See Michel et al. (2006) for a survey concerning bequest motives.

5 According to surveys conducted by Sallie Mae commissioned Gallup & Robinson, Inc. and reported by Miller (1997), most parents (92%) surveyed agree with the statement, “A college education is among the most important investment I will make for my child,” and “even with what it costs today, college is still a good investment.” The importance placed on their children’s education is confirmed, when 31% of the parents report that children’s college education is their highest financial priority, second only to the 38% of parents who indicate everyday budget is their first financial priority and almost twice the 17% of parents who name retirement as their first priority.

6 In estimates made by Gertler and Glewwe (1990), it was found that rural Peruvian households at all income levels are willing to pay fees high enough to cover the operating costs of new schools in their villages.
We study an otherwise the same model as Boldrin and Montes (2005), Docquier et al. (2007) and Bishnu (2013), except for altruism, which follows from Barro (1974). Like Docquier et al. (2007) and Bishnu (2013), the social planner maximizes the sum of utilities over generations discounted by a social weight. In Boldrin and Montes (2005), Docquier et al. (2007) and Bishnu (2013), the suboptimality is based on the comparison between the Diamond (1965) model and the corresponding planner problem. As a result, their suboptimality depends only on the social weight given to future generations by the social planner. In contrast, the suboptimality in our model is based on the comparison between the Barro (1974) model and the planner problem. Consequently, our suboptimality depends not only on the social weight given by the social planner but also on the altruism factor given to future descendants by the parent. Such a difference has important implications in policy issues to restore the optimality.

Recently, several papers considered altruism when analyzing economic issues. The closest to our paper is Alonso-Carrera et al. (2008), which examined the suboptimality of physical capital in a two-period OLG model with consumption externalities and altruism. Alonso-Carrera et al. (2008) envisaged social security programs and tax policies to restore the optimality of physical capital. Our paper is different, as we do not consider consumption externalities. Moreover, we study the suboptimality of human capital arising from altruism and investigate whether education subsidies or taxes can restore the optimality of human and physical capital, issues that cannot analyze in Alonso-Carrera et al. (2008).

Specifically, in our model, an agent’s utility includes children’s utilities discounted by an altruism factor, with a higher factor indicating a smaller discount given to children. Agents live for three periods. When young, they receive education but do not consume. They consume, save and pay children’s education when working (middle-aged), and consume and leave bequests for children when old (retired). While education accumulates human capital, savings pile up physical capital. Both types of capital increase final goods production. Focusing on steady state, we study the allocation in equilibrium, followed by the social optimum. We analyze the suboptimality of physical and human capital accumulation in equilibrium, and explore the policy to restore optimality with the emphasis on the role played by the extent of altruism.

We obtain three main findings as follows. First, agents always finance education for children, but they

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7 As Boldrin and Montes (2005) did not introduce a social planner, they had a restrictive definition of suboptimality.
8 Page (2003) empirically tested and found the existence of an operative bequest motive. Alonso-Carrera et al. (2007) examined how consumption habits and aspirations make the willingness to leave bequests stronger or weaker. Barnett et al. (2013) investigated how bequest-giving behavior gives rise to deviant generations, generations that do not leave a bequest having received an inheritance, and vice versa. Ihori et al. (2017) analyzed a model with altruism and liquidity constraints wherein children’s education costs are shared between children and parents with the share determined by parents.
9 In a two-period OLG model with consumption externalities, Abel (2005) studied the suboptimality of physical capital and analyzed social security programs and capital taxes to restore the optimality. Alonso-Carrera et al. (2008) extended the model of Abel (2005) to one with altruism.
may or may not leave bequests for children in equilibrium. Parents leave bequests if the altruism factor is larger than a threshold and thus, operative bequest motives, but do not leave bequests if otherwise and thus, inoperative bequest motives.

Next, the type of suboptimality depends on the extent of altruism. Under operative bequest motives, there are two regimes of suboptimality: both human and physical capital under-accumulate if the altruism factor is smaller than the social weight, but both over-accumulate if otherwise. By contrast, under inoperative bequest motives, there are three regimes of suboptimality: both human and physical capital under-accumulate if the social weight is large, both over-accumulate if the social weight is small, while human capital under-accumulates and physical capital over-accumulates if the social weight is in between.

Finally, to restore the optimality, we find that, under operative bequest motives, it is optimal to subsidize education and bequests financed by lump-sum taxes from the working age, if the altruism factor is smaller than the social weight, but optimal to tax education and bequests with tax revenues transferring to the working age, if otherwise. In this case, the basis for public education financing and public pensions is weak. In contrast, under inoperative bequest motives, the altruism factor is always smaller than the social weight and thus, it is optimal to subsidize education. Moreover, the socially optimal bequest is negative, indicating that the retired consume less than the optimum and thus, it is optimal to offer lump-sum transfers to the old. Education subsidies and public pensions are financed by taxing the working age a lump sum. In this case, public education financing and public pensions are necessary policies. Our quantitative exercise suggests that inoperative bequest motives are more in line with the data. Therefore, the rationale for public education financing and public pensions is strong.

These results contribute to the existing work in three perspectives. First, the degree of altruism plays a central role in determining the type of the suboptimality in our model. By contrast, the extent of altruism plays no role in Boldrin and Montes (2005), Docquier et al. (2007), and Bishnu (2013). In addition, our three regimes of suboptimality may appear like those in Docquier et al. (2007) and our two regimes of suboptimality may seem like those in Bishnu (2013). Yet, our regimes are determined by the extent of altruism, as opposed to the dependence on the social weight in Docquier et al. (2007) and Bishnu (2013).

Next, to restore optimality, our policy proposal is general that renders the policy suggested in the existing work as a special case. Our model suggests public education financing and public pensions at the same time only if the extent of altruism is so small that the bequest motive is inoperative. This result is different from not only Docquier et al. (2007) and Bishnu (2013), but also Boldrin and Montes (2005) who recommend the use of both public education financing and public pensions regardless of the operativeness of bequest motives. Besides, Docquier et al. (2007) proposed that the policy of public pension is not necessary if the social weight is high. Their proposal is reasonable only if the altruism
factor is so large that the bequest motive is operative. If otherwise, our model supports the policy of public pension even if the social weight is high. As well, Bishnu (2013) suggested not mix public pensions and education financing at the same time. His policy is not recommended here: under inoperative bequest motives, we support to use these two policies at the same time.

Finally, to restore optimality, Alonso-Carrea et al. (2008) suggested the policy of a positive capital tax and a pay-as-you-go pension program under inoperative bequest motives, and the policy of a positive capital tax and an estate tax under operative bequest motives. Like their suggestion, our model supports a public pension program when bequest motives are inoperative. However, we do not suggest the capital taxation, due to no consumption externalities. Moreover, we suggest the policy of public education financing if bequest motives are inoperative, and even if bequest motives are inoperative but the altruism factor is smaller than the social weight. The model of Alonso-Carrea et al. (2008) cannot analyze the education policy.

We organize this paper as follows. In Section 2, we set up an OLG model with altruism, with the competitive equilibrium and the social optimum analyzed in Subsections 2.1 and 2.2, respectively. In Section 3, we compare the competitive equilibrium with the social optimum. Section 4 analyzes optimal policies to restore optimality. Finally, we offer some concluding remarks in Section 5.

2. The model

Our model is otherwise the same as Docquier et al. (2007) and Bishnu (2013) except abstracting from human capital and consumption externalities and allowing for altruism. The economy comprises a sequence of overlapping generations, an initial old generation, and an infinitely lived government. Agents live for three periods of life. In the first period of life, agents do not consume but receive education. In the middle age (or working age) of life, agents work, consume, save and pay children’s education. In the old age of life, agents retire, consume and may leave bequests for children.

The generation that works in period $t$ is indexed by $t$. Thus, agents born in period $t-1$ are referred to as generation $t$. Let the population of generation $t$ be $N_t$, which grows at the rate of $n$, i.e., $N_t = nN_{t-1}$. Then, $N_{t-1} + N_t + N_{t+1}$ is the population size in period $t$. Denote by $e_{t-1}$ the education cost of an agent born in period $t-1$ (i.e., a generation $t$ agent), which is paid by parents of generation $t-1$. The education investment results in a level of human capital in period $t$ as described by the following process

$$h_t = \phi(e_{t-1}),$$

(1)

where $\phi(e)$ is a positive, strictly increasing and strictly concave function, that satisfies the Inada condition, namely, $\phi'(0)=\infty$ and $\phi'(\infty)=0$. Notice that, different from Docquier et al. (2007), our human capital technology is free from any effect of externality.
The economy has a final good $Y_t$, produced by a neoclassical technology $Y_t = F(K_t, H_t)$, where $K_t$ is aggregate physical capital and $H_t$ is aggregate human capital, with $F$ being homogeneous of degree 1 and satisfying the Inada condition. Denote capital per capita and human capital per capita at the beginning of period $t$ by $k_t = \frac{K_t}{N_t}$ and $h_t = \frac{H_t}{N_t}$, respectively. Moreover, denote the economy's physical human capital ratio at the beginning of period $t$ by $\tilde{k}_t \equiv \frac{K_t}{N_t} = \frac{k_t}{h_t}$, which will be referred to as effective capital per capita. Then, we rewrite $Y_t = H_t f(\tilde{k}_t)$, $f(\cdot) > 0$ and $f'(\cdot) > 0 > f''(\cdot)$. For simplicity, we assume that physical capital depreciates completely in a period. Assuming that the factor markets are competitive, in optimum, firms rent capital and hire labor to levels that equalize the factor price to the marginal product.

\[ R_t = R(\tilde{k}_t) \equiv f'(\tilde{k}_t), \quad R'(\tilde{k}_t) < 0, \tag{2a} \]

\[ w_t = w(\tilde{k}_t) \equiv f(\tilde{k}_t) - \tilde{k}_t f'(\tilde{k}_t), \quad w'(\tilde{k}_t) > 0, \tag{2b} \]

where $R_t$ is the gross return to capital in period $t$ and $w_t$ is the wage rate of raw labor in period $t$.

The felicity function is $u(c_t, d_{t+1})$, where $c_t$ and $d_{t+1}$ are consumption at the middle and old age, respectively. The felicity function is strictly increasing, strictly concave and additive in its two arguments, and satisfies the Inada condition. Notice that, different from Alonso-Carrera et al. (2008) and Bishnu (2013), our felicity function abstracts from consumption externalities. Parents are altruistic and may finance education and leave bequests for children. The altruism factor is $\beta \in [0,1)$, which stands for the extent of altruism. Following Barro (1974) and Abel (1987), the utility of the representative agent is

\[ V(e_{t-1}, b_t) = \max \left[ u(c_t, d_{t+1}) + \beta V(e_t, b_{t+1}) \right], \tag{3} \]

where $V(e_{t-1}, b_t)$ denote the lifetime utility of an agent of generation $t$, who receives education at a cost $e_{t-1}$ when young and inherits bequests $b_t$ when middle-aged.

A working-aged agent distributes labor income $w_t h_t$ and inheritance $b_t$ among consumption $c_t$, savings $s_t$ and children’s education $e_t$. When old, an agent receives the return from savings, which is allocated between consumption $d_{t+1}$ and bequests for children $b_{t+1}$. The budget constraints for an agent of generation $t$ when working and old are, respectively,

\[ w_t h_t + b_t = c_t + s_t + ne_t, \tag{4a} \]

\[ R_{t+1} s_t = d_{t+1} + nb_{t+1}. \tag{4b} \]

The problem of the representative agent is to maximize (3) subject to (1), (4a) and (4b) with $b_{t+1} \geq 0$, given $k_0$ and $h_0$, and taking as given $w_t$ and $R_{t+1}$ for all $t$. The optimal conditions are as follows. First, by the envelope theorem, we obtain
\[ \frac{\partial V}{\partial e_t} = u_{e_t}(c_t, d_{t+1})w_t\phi(e_{t+1}), \quad (5a) \]

\[ \frac{\partial V}{\partial b_t} = u_{e_t}(c_t, d_{t+1}). \quad (5b) \]

Moreover, the optimality conditions for \( s_t, e_t \) and \( b_{t+1} \), along with the use of (5a) and (5b), give

\[ u_{e_t}(c_t, d_{t+1}) = u_{d_{t+1}}(c_t, d_{t+1})R_{t+1}, \quad (6a) \]

\[ nu_{e_t}(c_t, d_{t+1}) \geq \beta \frac{\partial V}{\partial e_t} = \beta u_{e_t}(c_{t+1}, d_{t+2})w_{t+1}\phi(e_t), \quad (6b) \]

\[ nu_{d_{t+1}}(c_t, d_{t+1}) \geq \beta \frac{\partial V}{\partial b_t} = \beta u_{e_t}(c_{t+1}, d_{t+2}), \quad (6c) \]

where the equality in (6b) and (6c) holds if \( e_t > 0 \) and \( b_{t+1} > 0 \), respectively.

First, (6a) is the Euler equation for consumption between middle and old ages. Next, (6b) determines children's education. As \( \phi(e) \) satisfies the Inada condition, (6b) holds with an equality and thus, \( e_t \) is positive. Intuitively, a parent invests in \( n \) children's education to the level, when the loss in the marginal utility of consumption equals the gain in the (altruism factor) discounted marginal utility arising from children's consumption due to higher wage income. Finally, (6c) determines the bequest. If the bequest motive is operative, (6c) holds with an equality and \( b_t > 0 \). Then, a parent leaves bequests for children to the level when the decrease in the marginal utility of consumption equals the increase in the discounted future marginal utility arising from children's consumption. By contrast, if the bequest motive is not operative, (6c) holds with an inequality and then, \( b_t = 0 \).

To complete the model, we specify the goods market clearing condition. With capital depreciating fully in one period, aggregate savings in period \( t \) are the capital stock at the beginning of period \( t+1 \). Thus, the goods market clearing condition is

\[ s_t = nk_{t+1} + n\tilde{k}_{t+1}h_{t+1}. \quad (7) \]

### 2.1 Competitive equilibrium

This subsection analyzes the competitive equilibrium.

**Definition 1.** For given \( h_0 \) and \( k_0 \), a competitive equilibrium is the path \( \{c_t, d_t, k_t, s_t, e_t, b_t, h_t, w_t, R_t\}_{t=0}^\infty \) that satisfies firms’ optimization conditions (2a)-(2b), agents’ budget constraints and optimization conditions (4a)-(4b) and (6a)-(6c), human capital accumulation (1), the goods market clearing condition (7), and the transversality condition \( \lim_{t\to\infty} \beta^t u_{e_t}b_t = 0 \).
From now on, prices and allocations with a superscript $CE$ stand for those in competitive equilibrium. Using (2a) and (2b), we rewrite the decisions for savings and education in (6a) and (6b), respectively, as intertemporal and intergenerational consumption Euler equations,

$$\frac{u_{CE}}{u_{CE}} = f'(\tilde{c}_{i+1})$$ \hspace{1cm} (8a)

$$\frac{u_{CE}}{u_{CE}} = \frac{\beta}{n} \phi(e_{i}^{CE}) \mu(k_{i+1}^{CE})$$ \hspace{1cm} (8b)

in which the altruism factor $\beta$ affects consumption allocations across generations.

If condition (6c) is binding, we can use (6c) to rewrite the intertemporal consumption Euler equation in (8a) as $\frac{u_{CE}}{u_{CE}} = \frac{n}{\pi} f'(\tilde{k}_{i+1}^{CE})$. By using this expression and (2b), we can rewrite the intergenerational consumption Euler equation in (8b) as

$$\phi'(e_{i}^{CE}) = \frac{f'(\tilde{c}_{i}^{CE})}{f(k_{i+1}^{CE}) - k_{i+1}^{CE} f'(k_{i+1}^{CE})}$$ \hspace{1cm} (8c)

which is a relationship between next period’s effective capital $\tilde{k}_{i+1}^{CE}$ and this period’s education spending $e_{i}^{CE}$. As this period’s education spending determines next period’s human capital $h_{i+1}^{CE}$, (8c) is thus a relationship between next period’s effective capital and human capital.

In steady state, the allocations in competitive equilibrium are time-invariant, denoted by $\{c_{CE}, d_{CE}, k_{CE}, e_{CE}, b_{CE}, h_{CE}\}$. First, in the steady state, education investment for children in the intergenerational consumption Euler equation in (8b) is rewritten as

$$n = \beta \mu(\tilde{k}_{CE}) \phi'(e_{CE})$$ \hspace{1cm} (9a)

or, equivalently,

$$e_{CE} = e(\tilde{k}_{CE}) = (\phi')^{-1}\left(\frac{n}{\beta \mu(\tilde{k}_{CE})}\right), \text{ where } e'(\tilde{k}_{CE}) \cdot \frac{w'(-k_{CE}^{CE})}{\mu(k_{CE}^{CE})} > 0. \hspace{1cm} (9b)$$

Next, using (1), (2a), (2b), (4a), (4b), the intergenerational consumption Euler equation in (9b) enables us to write consumption when working and old, respectively, as functions of $e_{CE}$ and $\tilde{k}_{CE}$ as follows.$^{10}$

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$^{10}$ Note that $c_{CE}(\tilde{k}_{CE}, b_{CE}) = w(\tilde{k}_{CE}) \phi(e_{CE}) + b_{CE} - nk_{CE}^{CE} \phi(e_{CE}) - ne_{CE}^{CE}$ and $d_{CE}(\tilde{k}_{CE}, b_{CE}) = f'(\tilde{k}_{CE}) n k_{CE}^{CE} \phi(e_{CE}) - nb_{CE}^{CE}$, where $e_{CE} = e(\tilde{k}_{CE})$. In deriving these partial effects, following Alonso-Carrera et al. (2008) we assume that the negative effect of $\tilde{k}_{CE}$ on $c_{CE}$ dominates other effects and that the positive effect of $\tilde{k}_{CE}$ on $d_{CE}$ dominates other effects, so $\frac{\partial c_{CE}}{\partial \tilde{k}_{CE}} < 0$ and $\frac{\partial d_{CE}}{\partial \tilde{k}_{CE}} > 0$. Under a logarithmic utility and a Cobb–Douglas production function, conditions for $\frac{\partial c_{CE}}{\partial \tilde{k}_{CE}} < 0$ and $\frac{\partial d_{CE}}{\partial \tilde{k}_{CE}} > 0$ are easily met.
\[ c^{CE} = c^{CE}(\tilde{k}^{CE}, b^{CE}), \quad d^{CE} = d^{CE}(\tilde{k}^{CE}, b^{CE}), \]

where \( \frac{\partial c^{CE}}{\partial k^{CE}} < 0, \quad \frac{\partial c^{CE}}{\partial b^{CE}} > 0, \quad \frac{\partial d^{CE}}{\partial k^{CE}} > 0, \quad \frac{\partial d^{CE}}{\partial b^{CE}} < 0. \)

Moreover, in steady state, we rewrite the \textit{intertemporal} Euler equation in (8a) as \( \Gamma = u_c(c^{CE}, d^{CE}) - f'(\tilde{k}^{CE})u_c(c^{CE}, d^{CE}) = 0 \) and thus, \( \Gamma \) is a zero net marginal cost of savings (c.f. (6a)). Then, using (10), this \textit{intertemporal} Euler equation can be written as a function of \( \tilde{k}^{CE} \) and \( b^{CE} \) as follows:\(^{11}\)

\[ \Gamma(\tilde{k}^{CE}, b^{CE}) \equiv u_c(c(\tilde{k}^{CE}, b^{CE})) - f'(\tilde{k}^{CE})u_c(d(\tilde{k}^{CE}, b^{CE})) = 0. \quad (11) \]

Differentiating \( \Gamma(\tilde{k}^{CE}, b^{CE}) \) and using (10) yields \( \Gamma_k \equiv \frac{\partial \Gamma}{\partial k^{CE}} < 0 \) for all \( b^{CE} > 0 \) and \( \Gamma_b \equiv \frac{\partial \Gamma}{\partial b^{CE}} > 0 \) for all \( b^{CE} \geq 0 \). Intuitively, a larger bequest increases middle-aged agents’ consumption, decreases old agents’ consumption and thus, decreases the net marginal cost of savings. By contrast, a larger effective capital decreases middle-aged agents’ consumption, increases old agents’ consumption and thus, increases the net marginal cost of savings.

It suffices to impose the following assumption made by Abel (1987) and Alonso-Carrera et al. (2008).

\textbf{Assumption A.} There exists a unique \( \tilde{k}^0 > 0 \) satisfying \( \Gamma(\tilde{k}^{CE}, 0) = 0 \) with \( c(\tilde{k}^{CE}, 0) > 0, \quad d(\tilde{k}^{CE}, 0) > 0, \quad \text{and} \quad \Gamma_k(\tilde{k}^0, 0) > 0. \)

It is clear that properties \( \frac{\partial c^{CE}}{\partial k^{CE}} < 0 \) and \( \frac{\partial c^{CE}}{\partial b^{CE}} > 0 \) in (10) ensure that Assumption A is met. This assumption posits that, even with a zero bequest, there exists a positive effective capital \( \tilde{k}^{CE} \) consistent with positive consumption of middle-aged and old agents in the steady state. Moreover, even with a zero bequest, \( \Gamma_k(\tilde{k}^{CE}, 0) > 0 \) means that \( \Gamma(\tilde{k}^{CE}, 0) \) has a positive slope with respect to effective capital. Note that Assumption A implies \( \Gamma(\tilde{k}^{CE}, b^{CE}) > 0 \) for \( \tilde{k}^{CE} > \tilde{k}^0 \) and \( \Gamma(\tilde{k}^{CE}, b^{CE}) < 0 \) for \( \tilde{k}^{CE} < \tilde{k}^0 \). However, the bequest motive may or may not be operative and thus, \( b^{CE} \) may be positive or zero.

To see when the bequest motive is operative or inoperative, we substitute the binding steady-state condition for bequest choices (6c) \( u_{f_{CE}} = \frac{\theta}{n} u_{f_{CE}} \) into the condition of a zero net marginal cost of savings (11), and then evaluate it at a zero bequest. According to Assumption A, then there exists a threshold value of the altruism factor, denoted by \( \tilde{\beta} \), consistent with a zero bequest as follows:

\[ \tilde{\beta} = \frac{n}{f'(k^0)}. \]

\(^{11}\) Note that \( \Gamma(\tilde{k}^{CE}, b^{CE}) \) is defined for \( \tilde{k}^{CE} > 0 \) and \( b^{CE} > 0 \) such that \( c(\tilde{k}^{CE}, b^{CE}) > 0 \) and \( d(\tilde{k}^{CE}, b^{CE}) > 0 \).
As a result, if $\beta > \bar{\beta}$, the bequest motive is operative. Intuitively, when parents care for their children at a degree higher than the threshold value, they leave bequests for children. By contrast, if $\beta < \bar{\beta}$, the bequest motive is inoperative. Then, parents care for their children at a degree less than the threshold value, and they do not leave bequests.$^{12}$ Below, we derive steady-state conditions under the cases of operative and inoperative bequest motives, respectively.

**Case 1. Operative bequest motives**

In this case, the altruism factor $\beta$ is larger than the threshold value $\bar{\beta}$. First, the intertemporal consumption Euler equation (8a) in the steady state, accompanied by (6c) with an equality, gives

$$n = \beta f'(\tilde{k}^{CE}).$$

(12a)

Thus, a parent leaves bequests $b^{CE} > 0$ to the level, wherein the marginal cost of leaving bequests to $n$ children equals the altruism-factor discounted marginal product of effective capital resulting from saving bequests for children.

Next, the intergenerational consumption Euler equation (9a), which gives education investment for children, accompanied by the use of (12a), yields

$$w(\tilde{k}^{CE})\phi'(e^{CE}) = f'(\tilde{k}^{CE}).$$

This condition equalizes effective wage for children and the marginal product of effective capital resulting from leaving bequests for children. With the use of (2b), we rewrite this condition as

$$[f(\tilde{k}^{CE}) - \tilde{k}^{CE} f'(\tilde{k}^{CE})]\phi'(e^{CE}) = f'(\tilde{k}^{CE}),$$

(12b)

and it is easy to show that $\frac{de^{CE}}{dk} > 0$. $^{13}$

Thus, when bequest motives are operative, intertemporal and intertemporal conditions in (12a) and (12b) together characterize steady-state effective capital and education investment, and thus human capital, in competitive equilibrium.

**Case 2. Inoperative bequest motives**

In this case, the altruism factor $\beta$ is smaller than the threshold value $\bar{\beta}$ . First, the intertemporal

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$^{12}$ The threshold value of the altruism factor is similar to the one obtained in Weil (1987). Here, as human capital investment is introduced, the threshold value of the altruism factor is changed through the effect on physical and human capital; that is via the effect on effective capital per capita $\tilde{k}^0$.

$^{13}$ Although a similar condition is also obtained in Bishnu (2013), agents borrow their own education costs in Bishnu (2013), while parents pay for children's education costs and have operative bequest motives in our model. As a result, the marginal product of effective capital is due to the gross loan interest payment in Bishnu (2013), but it is due to the decrease in a parent's savings resulting from leaving bequests for children in our model. As will be seen below, when parents' bequest motives are inoperative, this condition does not hold.
consumption Euler equation (8a) in the steady state, accompanied by (6c) with an inequality, gives
\[ n > \beta f'(\bar{k}^{CE}). \] (13a)

As the marginal cost of leaving bequests to \( n \) children is larger than the altruism-factor discounted marginal product of effective capital resulting from saving bequests for children, a parent leaves no bequests to children, and thus \( b^{CE} = 0 \).

Next, with (13a), the intergenerational condition in (9a) yields \( w(\bar{k}^{CE})\phi'(e^{CE}) > f'(\bar{k}^{CE}) \). Thus, effective wage for children is larger than the marginal product of effective capital resulting from saving bequests for children. Hence, under inoperative bequest motives, a parent pays for education for children but leaves no bequests for children. With the use of (2b), we rewrite this intergenerational condition as
\[ f'(\bar{k}^{CE}) - \bar{k}^{CE} f''(\bar{k}^{CE}) \phi'(e^{CE}) > f'(\bar{k}^{CE}). \] (13b)

Thus, when bequest motives are inoperative, intertemporal and intertemporal conditions in (13a) and (13b) together characterize steady-state effective capital and education investment, and thus human capital, in competitive equilibrium.

2.2 Social planner’s allocations

This subsection studies the social planner’s problem. The social planner’s resource constraint is
\[ h_t f(\bar{k}_t) = c_t + \frac{d}{n} + n(e_t + k_{t,t+1}), \] (14)
and the human capital accumulation in (1) for all \( t \).

The social planner’s utility is the sum of utilities over generations with a social weight \( \lambda \in (0, 1) \) given to future generations. Like Docquier et al. (2007) and Bishnu (2013), the social weight \( \lambda \) is a discount factor, that a social planner attaches to future generations. A high \( \lambda \) indicates a smaller social discount rate, in that the social planner devalues less of future generations. In the analysis that follows, a social planner with a different social weight \( \lambda \) may be interpreted as a different social planner indexed by \( \lambda \).

The social planner’s problem is to maximize the discounted sum of utilities over generations, subject to the human capital accumulation (1) and resource constraint (14) for all \( t \). The Lagrangian of the social planner’s problem is thus as follows:
\[ L = \sum_{t=0}^{\infty} \lambda^t \left\{ u(c_t, d_t, e_t, k_{t,t+1}) + q_t \left[ \phi'(e_{t-1}) f'(\bar{k}_t) - c_t - \frac{d}{n} - n(e_t + k_{t,t+1}) \right] \right\}, \]
where \( \lambda^t q_t \) is the multiplier associated with the resource constraint in period \( t \).

The first-order conditions with respect to \( \{c_t, d_t, e_t, k_{t,t+1}\}_{t=0}^{\infty} \) are, respectively,
\[ \lambda u_{c_t} - \lambda^t q_t = 0, \] (15a)
\[
\begin{align*}
    u_{c_{i+1}} - \lambda \frac{q_i}{n} &= 0, \\
    \lambda q_{i+1} \phi'(c_{i+1}^{SP}) \left( f(\bar{k}_{i+1}^{SP}) - f'(\bar{k}_{i+1}^{SP})\bar{k}_{i+1}^{SP} \right) - n q_i &= 0, \\
    \lambda q_{i+1} f'(\bar{k}_{i+1}^{SP}) - n q_i &= 0,
\end{align*}
\]

where superscript \( SP \) is used to stand for the social planner's outcome.

**Definition 2.** For a given social weight \( \lambda \) and initial stock \( h_0 \) and \( k_0 \), the social optimum is the path \( \{c_i^{SP}, d_i^{SP}, \bar{k}_i^{SP}, e_i^{SP}, h_i^{SP}, q_i^{SP}\}_{t=0}^{\infty} \) that satisfies the human capital accumulation (1), the resource constraint (14), the first-order conditions (15a)-(15d), and the transversality condition \( \lim_{t \to \infty} \bar{q}_t \bar{k}_t^{SP} = 0 \).

From (15a) and (15b), we obtain the social planner's *intratemporal* consumption Euler equation.

\[
\frac{u_{c_i^{SP}}}{u_{d_i^{SP}}} = \frac{n}{\lambda}.
\]

With the help of (16), we can use (15a) and (15d) to derive the social planner’s *interperiodal* and *intergenerational* consumption Euler equations, respectively, as follows.

\[
\begin{align*}
    \frac{u_{c_i^{SP}}}{u_{d_i^{SP}}} &= f'(\bar{k}_{i+1}^{SP}), \\
    \frac{u_{c_i^{SP}}}{u_{d_i^{SP}}} &= \frac{\lambda}{n} f'(\bar{k}_{i+1}^{SP}),
\end{align*}
\]

in which the social weight \( \lambda \) affects the planner’s consumption allocations across generations.

Moreover, with the use of (15a) and (15c), we can also rewrite the *intergenerational* consumption Euler equation in (17d) as

\[
\phi'(c_{i+1}^{SP}) = \frac{f'(\bar{k}_{i+1}^{SP})}{f(\bar{k}_{i+1}^{SP}) - f'(\bar{k}_{i+1}^{SP})\bar{k}_{i+1}^{SP}},
\]

which relates next period’s effective capital \( \bar{k}_{i+1}^{SP} \) to this period’s education investment \( e_{i+1}^{SP} \) and thus, is a relationship between next period’s effective capital and human capital \( h_{i+1}^{SP} \).

In the steady state, the allocations in the social optimum are time-invariant, denoted by \( \{c^{CE}, d^{CE}, \bar{k}^{CE}, e^{CE}, h^{CE}\} \). First, in the steady state, *intratemporal* and *interperiodal* consumption Euler equations (16) and (17a) are equal, leading to \( \frac{c_{i+1}^{SP}}{d_{i+1}^{SP}} = \frac{n}{\lambda} = f'(\bar{k}_i^{SP}) \) and thus,

\[
n = \lambda f'(\bar{k}_i^{SP}),
\]
which is the modified golden rule condition. This condition determines the golden rule effective capital per capita $\bar{k}^{sp}$. Thus, if a social planner assigns a larger social weight to future generations (a larger $\lambda$), there would be a larger optimal effective capital per capita in the long run.

Moreover, in a steady state, the intergenerational consumption Euler equation (17c) is

$$[f'(\bar{k}^{sp}) - f'(\bar{k}^{sp})\bar{k}^{sp}]\phi'(\bar{e}^{sp}) = f'(\bar{k}^{sp}),$$

(18b)

and it is easy to show that $\frac{d\bar{e}^{sp}}{de} > 0$.

Thus, (18a) and (18b) together characterize steady-state effective capital and education investment in social optimum.

3. Competitive Equilibrium vs Social Optimum

This section analyzes whether or not the allocations of physical and human capital in competitive equilibrium are social optimal. We focus on the steady state. On the one hand, the altruism factor affects the allocation of savings, education investment and bequests in the equilibrium. On the other hand, the social weight plays a central role in determining the social optimal allocation. Therefore, the altruism factor and the social weight affect the suboptimality concerning physical and human capital accumulation in equilibrium. We start with the case of operative bequest motive, followed by the case of inoperative bequest motive.

3.1 Suboptimality of the equilibrium when the bequest motive is operative

Under operative bequest motives, in the steady state, competitive equilibrium conditions are (12a) and (12b), and social optimum conditions are (18a) and (18b). When comparing (12a) and (18a), we note that whether or not $\lambda$ is equal to $\beta$ influences whether or not $\bar{k}^{ce}$ is equal to $\bar{k}^{sp}$ and thus, whether or not physical capital accumulation in equilibrium $\bar{k}^{ce}$ is suboptimal. In turn, in comparing (12b) and (18b), whether or not $\bar{k}^{ce}$ is equal to $\bar{k}^{sp}$ affects whether or not $\bar{e}^{ce}$ is equal to $\bar{e}^{sp}$ and thus, whether or not human capital in equilibrium $\bar{h}^{ce}$ is suboptimal. Therefore, the values of the altruism factor and the social weight crucially affect the suboptimality of the equilibrium.

Let $\eta^i_{k,e}, j=k,h, i=CE, SP$, denote the percentage change in physical capital and human capital when the education expenditure is increased by one percent. That is, $\eta^i_{k,e} = \frac{\partial k^i}{\partial e^i}$ and $\eta^i_{h,e} = \frac{\partial h^i}{\partial e^i}$.

We establish the following result.

**Lemma 1.** For $i=CE, SP$, $\eta^i_{k,e} > \eta^i_{h,e}$ holds. For any $\lambda$, $\frac{d\bar{e}^{sp}}{d\lambda} > 0$ and $\frac{d\bar{e}^{sp}}{d\lambda} > 0$.

In words, the first part says that if the education expenditure increases by one percent, the
proportional increase of physical capital per capita is larger than the proportional increase of human capital per capita. The second part reveals that both effective capital per capita and education investment in the social optimum increase in the social weight.\textsuperscript{14} Intuitively, as the social planner increases the weight for future generations, the socially optimal physical and human capital are larger.

Next, we establish a uniqueness result.

**Lemma 2.** Under operative bequest motives, if there exists a $\lambda = \lambda^*$ such that $\tilde{k}^{SP}(\lambda^*) = \tilde{k}^{CE}(\beta)$ holds, then $e^{SP}(\lambda^*) = e^{CE}(\beta)$ holds, and vice versa.

As the expression of (12b) in equilibrium is the same as that of (18b) in optimum, Lemma 2 easily follows from the fact that if $\lambda = \beta$, then (12a) in equilibrium and (18a) in optimum are the same and thus, physical capital and human capital in equilibrium are the same as those in optimum. However, in general, $\lambda \neq \beta$, and then (12a) in equilibrium and (18a) in optimum are different. Therefore, physical capital and human capital in equilibrium are different from those in optimum.

To see what the difference is, first, if $\lambda > \beta$, then (12a) and (18a) lead to $k^{CE}(\beta) < \tilde{k}^{SP}(\lambda)$, which, using (12b) and (18b), implies $e^{CE}(\beta) < e^{SP}(\lambda)$. Thus, we attain $h^{CE}(\beta) < h^{SP}(\lambda)$ and $k^{CE}(\beta) < k^{SP}(\lambda)$, according to Lemma 1.

Alternatively, if $\lambda < \beta$, then (12a)-(12b) and (18a)-(18b) give $k^{CE}(\beta) > \tilde{k}^{SP}(\lambda)$ and $e^{CE}(\beta) > e^{SP}(\lambda)$, which imply $h^{CE}(\beta) > h^{SP}(\lambda)$ and $k^{CE}(\beta) > k^{SP}(\lambda)$, according to Lemma 1.

As a result, we obtain $h^{CE}(\beta) < h^{SP}(\lambda)$ and $k^{CE}(\beta) < k^{SP}(\lambda)$ if $\lambda > \beta$, and $h^{CE}(\beta) > h^{SP}(\lambda)$ and $k^{CE}(\beta) > k^{SP}(\lambda)$ if $\lambda < \beta$. We summarize these results in the following proposition.

**Proposition 1.** Under operative bequest motives, the steady-state levels of physical and human capital accumulation in competitive equilibrium are larger than the social optimum if the social weight is larger than the altruism factor, but are smaller than the social optimum if otherwise.

Intuition goes as follows. When $\lambda > \beta$, the social planner cares for future generations more than the patent. Then, physical and human capital in equilibrium are under-accumulated, as compared to the social

\textsuperscript{14} The first part of lemma 1 is easily shown, if we differentiate (12b) with respect to $k^{CE}$ and $e^{CE}$ for a given $h^{CE}$, and also differentiate (12b) with respect to $h^{CE}$ and $e^{CE}$ for a given $k^{CE}$, and then we obtain $\eta^{CE}_{k,e} > \eta^{CE}_{h,e}$. Moreover, if we differentiate (18b), we get $\eta^{SP}_{k,e} > \eta^{SP}_{h,e}$. Hence, $\eta^{SP}_{k,e} > \eta^{SP}_{h,e}$, $i = CE, SP$. For the second part, if we differentiate (18a), we obtain $\frac{dk^{SP}}{dx} > 0$. Moreover, differentiating (18b) yields $\frac{dh^{SP}}{dx} > 0$, implying $\frac{dk^{SP}}{dx} > 0$. 14
optimum. By contrast, if \( \lambda < \beta \), the social planner cares for future generations less than the patent. Then, physical and human capital in equilibrium are over-accumulated, as compared to the social optimum.

Figure 1 illustrates the regions of the social weight in the horizontal axis and the altruism factor in the vertical axis that characterize the suboptimality of physical and human capital accumulation in equilibrium. In the vertical axis, the bequest motive is operative in the section where the altruism factor is above the threshold \( (\beta > \bar{\beta}) \). This section has two regimes. In Regime I, wherein the social weight is larger than the altruism factor \((\lambda > \beta)\), physical and human capital under-accumulate in competitive equilibrium, as compared to the social optimum. By contrast, in Regime II, wherein the social weight is smaller than the altruism factor, physical and human capital over-accumulate in competitive equilibrium, as compared to the social optimum.

3.2 Suboptimality of the equilibrium when the bequest motive is inoperative

This subsection analyzes the situation when \( \beta < \bar{\beta} \), and thus the bequest motive is inoperative. In this environment, (6c) is not binding and thus, \( b=0 \). The allocation of physical capital and human capital investment in the competitive equilibrium in steady state satisfies (13a) and (13b), which is compared with (18a) and (18b) in the social optimum.

When the bequest motive is inoperative, Lemma 1 remains hold true.\(^{15}\) However, Lemma 2 does not hold.\(^{16}\) Notwithstanding the foregoing, with \( \phi'(e) < 0 \), comparing (13a) with (18a) and (13b) with (18b) easily gives the following lemma.

Lemma 3. If there exists a \( \lambda_1 \) such that \( \bar{k}^{SP}(\beta + \lambda_1) = \bar{k}^{CE}(\beta) \) holds, then \( e^{SP}(\beta + \lambda_1) > e^{CE}(\beta) \) holds; and if there exists a \( \lambda_2 \) such that \( e^{SP}(\beta + \lambda_2) = e^{CE}(\beta) \) holds, then \( \bar{k}^{SP}(\beta + \lambda_2) < \bar{k}^{CE}(\beta) \) holds. In addition, \( \lambda_1 > \lambda_2 \).

We now analyze the suboptimality of physical and human capital accumulation in the competitive equilibrium. Under \( \beta < \bar{\beta} \), Lemma 3 indicates that there exist \( \lambda_1 \) and \( \lambda_2 \), with \( \lambda_1 > \lambda_2 \), such that \( \bar{k}^{SP}(\lambda + \beta_1) = \bar{k}^{CE}(\beta) \), \( \bar{k}^{SP}(\lambda_2 + \beta) < \bar{k}^{CE}(\beta) \), and \( h^{SP}(\lambda_2 + \beta) = h^{CE}(\beta) \). As a result, \( k^{SP}(\lambda_1 + \beta) > k^{CE}(\beta) > k^{SP}(\lambda_2 + \beta) \). Moreover, as \( k^{SP} \) is continuous in \( \lambda \), it follows that there exists a \( \lambda_* \in (\lambda_2, \lambda_1) \) such

\(^{15}\) By comparing the education investment condition in (9a) in the competitive equilibrium with (18b) in the social optimum, it is straightforward to verify that \( \frac{\delta e}{\delta \lambda} > 0 \), \( i = CE, SP \), and thus \( \eta_0 > \eta_* \) holds in the steady state.

\(^{16}\) To see this, if we compare (13a) in the competitive equilibrium with (18a) in the social optimum, we obtain \( \bar{k}^{CE} > \bar{k}^{SP} \) even when \( \lambda = \beta \). Moreover, if we compare (13b) with (18b), we find that the equality results of \( \bar{k}^{SP}(\lambda^*) = \bar{k}^{CE}(\beta) \) and \( e^{SP}(\lambda^*) = e^{CE}(\beta) \) in Lemma 2 do not hold.
that \( k^{SP}(\lambda + \beta) = k^{CE}(\beta) \).

Therefore, we have established the following results.

(i) \( k^{CE} < h^{SP} \) and \( k^{CE} < k^{SP} \) for all \( \lambda < \lambda_2 + \beta \);

(ii) \( k^{CE} > h^{SP} \) and \( k^{CE} > k^{SP} \) for all \( \lambda > \lambda_2 + \beta \);

(iii) \( k^{CE} < h^{SP} \) and \( k^{CE} > k^{SP} \) for all \( \lambda \in (\lambda_2 + \beta, \lambda_3 + \beta) \).

We summarize these results in the following proposition.

**Proposition 2.** With an inoperative bequest motive, as compared to the social optimum, physical and human capital in the competitive equilibrium in steady state under-accumulate if the social weight is large, over-accumulate if the social weight is small, and physical capital over-accumulates while human capital under-accumulates if the social weight is in between.

We illustrate the suboptimality of physical and human capital accumulation in equilibrium in the section where the altruism factor is below the threshold (\( \beta < \beta_0 \)) in Figure 1. This section has three regimes. In Regime III, where \( \lambda < \lambda_2 + \beta \), as the social weight is sufficiently small, physical and human capital accumulation over-accumulate in competitive equilibrium, as compared to the social optimum. By contrast, in Regime V, where \( \lambda > \lambda_3 + \beta \), as the social weight is sufficiently large, physical and human capital accumulation under-accumulate in competitive equilibrium, as compared to the social optimum. Finally, in Regime IV, where the social weight is in between, physical capital over-accumulates, while human capital under-accumulates in competitive equilibrium, as compared to the social optimum.

It is interesting to compare our findings of suboptimality with the existing literature. First, our two regimes (Regimes I and II) separating over- or under-accumulation of physical and human capital are reminiscent of those in an otherwise the same model in Bishnu (2013). The key difference is that our result emerges only when the altruism factor is above the threshold and thus, the bequest motive is operative. Moreover, our three regimes (Regimes III, IV and V) separating over- or under-accumulation of physical and human capital seem similar to those in an otherwise the same model except with human capital externalities and without altruism in Docquier et al. (2007). The central difference is that our result arises only when the altruism factor is below the threshold and thus, the bequest motive is in operative. Clearly, the extent of altruism is important in determining the regimes of suboptimality. As a result, it is apparent that the extent of altruism will also affect the set of policy instruments to restore optimality, an issue to which we turn in the next section.

4. **Optimal Policy**

In this section, we characterize the optimal policy in order to implement the allocation in equilibrium.
as a social optimum. In the following, we will abstract from capital income taxes, labor income taxes, and consumption taxes.\footnote{Without consumption externalities, capital income and consumption taxes play no role. Moreover, it may be easily shown that optimal labor income taxes are zero.} We investigate which instruments of education subsidies, estate taxes and intergenerational transfers can restore optimality and, in particular, when both policies of education subsidies and public pension benefits are necessary.

The government levies an estate tax on bequests, a tax on education investment, and a lump-sum tax in periods of work and retirement. With these taxes, budget constraints of an individual of generation $t$ in periods of work and retirement in (4a) and (4b) are now modified, respectively, as

\[ w_i s_t - T_i^m + (1 - \tau_i^b) b_t = c_t + s_t + (1 + \tau_i^e) ne_t, \]

\[ R_{t+1} s_t - T_{t+1}^o = d_{t+1} + nb_{t+1}, \]

where $\tau_i^b$ is the estate tax rate on bequests, $\tau_i^e$ is the tax rate on education investment, $T_i^m$ is the lump-sum tax in periods of work and $T_{t+1}^o$ is the lump-sum tax in periods of retirement.

The government budget constraint in period $t$ is\footnote{Let the population of the working age in period $t$ be unity. Then, the population for the old and the young in period $t$ is $1/n$ and $n$, respectively.}

\[ T_i^m + \tau_i^b b_t + \frac{T_{t+1}^o}{n} + \tau_i^e n e_t = 0. \]

The optimization problem of the agent of generation $t$ is to maximize (3) subject to (1), (19a) and (19b), given $k_0$ and $h_0$ and taking all taxes and $w_t$ and $R_{t+1}$ for all $t$ as exogenously given. The optimality conditions for $s_t$, $e_t$ and $b_{t+1}$ are as follows.

\[ u_{t_0}(c_t, d_{t+1}) = u_{d_{t+1}}(c_t, d_{t+1}) R_{t+1}, \]

\[ (1 + \tau_i^e) u_{t_0}(c_t, d_{t+1}) = \beta u_{t_1}(c_{t+1}, d_{t+2}) w_{t+1} \phi(e_t), \]

\[ n u_{d_{t+1}}(c_t, d_{t+1}) \geq \beta u_{t_1}(c_{t+1}, d_{t+2})(1 - \tau_i^b), \]

where the equality in (21c) holds if $b_{t+1} > 0$.

**Definition 3.** For given $h_0$ and $k_0$ and a given path of tax policies $\{T_i^m, T_{t+1}^o, \tau_i^e, \tau_i^b\}_{t=0}^\infty$, a competitive equilibrium is a path $\{c_t, d_t, k_t, s_t, e_t, b_t, h_t, w_t, R_t\}_{t=0}^\infty$ that satisfies firms’ optimization (2a)-(2b), the budget constraint (19a)-(19b), agents’ optimizations (21a)-(21c), the human capital accumulation (1), the goods market clearing condition (7), and the transversality condition $\lim_{t \to \infty} \beta^t u_{t_0} b_t = 0$.

Let a value and a policy with a “crescent” symbol on their top denote an optimal value and an
optimal policy in competitive equilibrium. Then, the path of the optimal policy is \( \{\hat{T}_t^n, \hat{T}_t^s, \hat{r}_t^b, \hat{r}_t^e\}_{t=0}^\infty \). The optimal policy is such that the tax-corrected equilibrium path \( \{\hat{c}_t^s, \hat{d}_t^s, \hat{k}_t^s, \hat{e}_t^s\} \) coincides with the socially optimal path \( \{c_t^{sp}, d_t^{sp}, k_t^{sp}, e_t^{sp}\} \). Following Alonso-Carrera et al. (2008), if we substitute savings in the budget constraint of the old in \((4b)\) into the goods market condition \((7)\), the optimal path of bequests implied by the planner’s solution is

\[
b_t^{sp} = f'(\hat{k}_t^{sp})\hat{k}_t^{sp} - \frac{d_t^{sp}}{n}. \tag{22}\]

A positive optimal bequest implies that the bequest motive is operative along an equilibrium path that achieves the first best solution, while a negative optimal bequest implies that the bequest motive is inoperative along this equilibrium path.

Finally, using factor returns in \((2a)\) and \((2b)\), we rewrite \((21a)-(21c)\), respectively, as

\[
\frac{u_{c_t}}{u_{d_t}} = f'(\hat{k}_{t+1}), \tag{23a}\]

\[
\frac{u_{c_t}}{u_{e_t}} = \frac{\beta}{n(1+\tau_t^b)} \left[ f'(\hat{k}_{t+1}) - \hat{k}_{t+1}f'(\hat{\hat{k}}_{t+1}) \right] \phi'(e_t), \tag{23b}\]

\[
\frac{u_{d_t}}{u_{d_t}} = \frac{n}{\beta(1-\tau_t^b)}. \tag{23c}\]

We will characterize the optimal tax policy by distinguishing the case of operative bequest motives from that of inoperative bequest motives.

### 4.1 Operative bequest motives

In this case, the equilibrium path associated to the optimal policy is characterized by \((19a)-(19b)\), \((20)\) and \((23a)-(23c)\). We investigate which policy instruments should be used in order for the allocation in the competitive equilibrium to achieve the social optimum.

First, if we combine \((23a)-(23c)\), we get

\[
\phi'(e_t) = \frac{(1+\tau_t^b)(1-\tau_t^e)}{f(k_{t+1})-k_{t+1}f'(k_{t+1})} f'(\hat{k}_{t+1}), \tag{24}\]

which is like \((8c)\), an expression of an intergenerational consumption Euler equation.\(^{19}\)

Next, if we compare social optimum conditions \((16)\), \((17a)\), \((17b)\) and \((17c)\) with tax-corrected competitive equilibrium conditions \((23c)\), \((23a)\), \((23b)\) and \((24)\), respectively, then we obtain \( \hat{r}_t^e = \frac{\beta}{\tau} - 1 \)

\[^{19}\text{If we substitute } u_{c_t} \text{ in (23a) into (23b), we get an expression of } \frac{k_{t+1}}{k_{t+1}}. \text{ If we backward this expression by one period and equate it with the right-hand-side of (23c), we give (23d).}\]
and \( \tau_i^e = 1 - \frac{\beta}{\lambda} \).

Moreover, using these tax rates along with period-t budget constraints at the working age (19a) and the old (19b), and the government budget constraint (20), we obtain lump-sum taxes for the old \( T_o = 0 \) and the working age \( \bar{T}_w = \bar{T}_w^{m1} + \bar{T}_w^{m2} \), with the portion \( \bar{T}_w^{m1} = -\tau^e \cdot n \cdot e \) being the tax (or subsidy) to finance education subsidy and \( \bar{T}_w^{m2} = -\tau^b \cdot b \) being the tax (or subsidy) to fund bequest subsidies.

To summarize the optimal tax policy, we obtain

**Proposition 4.** When the bequest motive is operative, the optimal tax policy is

\[
\tau_i^e = \frac{\beta}{\lambda} - 1, \quad \tau_i^b = 1 - \frac{\beta}{\lambda}, \quad \bar{T}_w = 0, \quad \text{and} \quad \bar{T}_w^{m1} = -\tau_i^e \cdot n \cdot e, \quad \bar{T}_w^{m2} = -\tau_i^b \cdot b.
\]

We illustrate these optimal policies in the top panel of Table 1. In Regime I, when the altruism factor is small than the social weight (\( \beta < \lambda \)), it is optimal to subsidize both education and bequests \( (\tau_i^e < 0, \tau_i^b < 0) \). These subsidies are financed by taxing the working age a lump sum \( (\bar{T}_w > 0) \). However, it is optimal to tax both education and bequests \( (\tau_i^e > 0, \tau_i^b > 0) \) when the altruism factor is larger than the social weight in Regime II. These tax revenues are transferred as a lump sum to the working age \( \bar{T}_w < 0 \).

Intuitively, education and bequests are parents’ transfers in kind and in goods, respectively. If \( \beta < \lambda \) and the parent cares for descendants less than the social planner, physical and human capital and bequests in equilibrium are all smaller than the social optimum. As capital taxes are against dynamic efficiency, it is optimal to subsidize both education and bequests in order for education investment and bequests to increase to their socially optimal levels. To balance the government budget, it is optimal to tax the working agent a lump sum equal to the sum of subsidies to education and bequests.

By contrast, if \( \beta > \lambda \) and the parent cares for descendants more than the social planner, physical and human capital and bequests are all larger than the social optimum. Then, it is optimal to levy education and estate taxes, in order for education investment and bequests to decrease to their socially optimal levels. To balance the government budget, it is optimal to transfer to the working agent a lump sum equal to the sum of education and estate taxes.

### 4.2 Inoperative bequest motives

In this case, (21c) is not binding and \( b_o = 0 \). The equilibrium path associated to the optimal tax policy
is characterized by (19a)-(19b), (20), (23a)-(23b), and $b_t=0$. Then, the counterpart of (24) is:

$$\phi'(e_t) > \frac{(1 + \varepsilon^h)(1 - \varepsilon^h) f'(\hat{k}_{t+1})}{f'(\hat{k}_{t+1}) - \hat{k}_{t+1} f'(\hat{k}_{t+1})}. \tag{25a}$$

The optimal policy is then analyzed by comparing the social optimum conditions (17a)-(17c) with (23a), (23b) and (25a), respectively. First, since $b_t=0$, the bequest tax is useless and thus, $\varepsilon^h = 0$. Next, comparing (23b) with (17b), with the use of (17c), we obtain $\varepsilon^h = \frac{\beta}{2} - 1$, which is the same as the case of operative bequest motives. Moreover, comparing (25a) with the intergenerational consumption Euler equation for the social optimum in (17c) indicates $\varepsilon^h < 0$. This implies that, when the bequest motive is inoperative, $\beta$ is always smaller than $\lambda$.

The suboptimality due to inoperative bequest motives is resolved through lump-sum taxes. By using the return to capital $R_t = f'(\hat{k}_t)$ in (2a) and capital accumulation $s_{t+1} = n k_t = n \hat{k}_t h_t$ in (7), along with $b_t=0$, the budget constraint of the old in (19b) gives the consumption of the old in period $t$ as follows.

$$d_t = n f'(\hat{k}_t) \hat{k}_t - T^o_t \tag{25b}$$

Then, we replicate the tax-corrected market allocation of consumption of the old, and physical and human capital in (25b) and the social optimum in (22), namely, $d_t = d_{t}^{SP}$, $\hat{k}_t = \hat{k}_{t}^{SP}$ and $h_t = h_{t}^{SP}$. We find that it is optimal to transfer a lump sum to an old, namely, $\varepsilon^h > 0$. Moreover, by using the government budget constraint in (20), it is optimal to tax the working age a lump sum, namely, $\bar{T}^o_t = \bar{T}^{o3} + \bar{T}^{o4}$, in which $\bar{T}^{o3} = -n \hat{c}_t \varepsilon^h > 0$ is to finance education subsidies, while $\bar{T}^{o4} = -b_t^{SP} > 0$ is to finance pensions.

To summarize the optimal tax policy, we obtain

**Proposition 5.** When the bequest motive is inoperative, the optimal tax policy is $\varepsilon^h = \frac{\beta}{2} - 1 < 0$, $\bar{T}_t = nb_t^{SP} < 0$ and $\bar{T}_t = -n \hat{c}_t \varepsilon^h - b_t^{SP} > 0$.

We illustrate these optimal policies in the bottom panel of Table 1. Under inoperative bequest motives, there is zero bequest, but the motive of transfer in kind remains at work. Moreover, under inoperative bequest motives, the altruism factor is always smaller than the social weight. As a result, different from the case of operative bequest motives, in all Regimes III, IV and V, it is optimal to

---

20 If we substitute $u_t$ in (23a) into (23b), we get an expression of $\frac{\xi_{u_t}}{k_{u_t}}$. If we backward the expression by one period backward and compare it with (23c), we obtain (25a).

21 Note that (22) can be rewritten as $d_{t}^{SP} = n f'(\hat{k}_t) \hat{k}_{t}^{SP} h_{t}^{SP} - nb_t^{SP}$.
subsidize education \((\hat{r}_t < 0)\), financed by taxing a lump sum \((\hat{T}_w^{ext} = -\hat{r}_t ne > 0)\) from the working age. Further, the socially optimal bequest is negative \((\hat{b}_t^{sep} < 0)\), indicating that retirees consume less than the social optimum. This case then calls for an intergenerational transfer in all Regimes III, IV and V. The transfer to retirees is in a lump-sum fashion like a public pension benefit (i.e., a negative lump-sum tax \(\hat{T}_w = nb_t^{sep} < 0\)), financed by taxing a lump sum \((\hat{T}_w^{ext} = -b_t^{sep} > 0)\) in periods of the working age.

### 4.3 Numerical analysis

This subsection carries out an analysis in order to numerically illustrate the optimal tax policy. Following Docquier et al. (2007) and Bishnu (2013), we consider the following functional forms for the preference, the human capital technology, and the production technology.

\[
\begin{align*}
u(c_t, d_{t+1}) &= \ln c_t + \rho \ln d_{t+1}, \\
n(r_t) &= Be^{r_t}, \\
\phi(c_t) &= \chi c_t^{\alpha-
end{align*}
\]

Based on these parametric functional forms, in the Appendix we have derived the expression in order to calculate the threshold value of the altruism factor, which separates operative and inoperative bequest motives. Further, the Appendix has derived the expressions that enable us to compute the optimal taxes or subsidies as functions of the altruism factor \(\beta\) and the social weight \(\lambda\).

For parameters, we take the following values:

\[
\{A = 1, \alpha = \chi = 0.35, n = 1.1, \rho = 0.8\}
\]

We normalize the value of \(A\) to 1 for simplicity. We set \(\alpha = 0.35\), so the share of output in capital is about one third. We also set \(\chi = 0.35\), so the contribution of education investment to education output is the same as the contribution of capital in the final goods. Following Docquier et al. (2007), we assume one period (a cohort) represents 20 years. We also go along with Docquier et al. (2007) and use \(n = 1.1\) in order to match 0.5% annual growth rate of the population for a 20-year period. With reasonable annual rates of the time preference ranging from 0.5% to 2%, we set \(\rho = 0.8\) for a 20-year period. In deriving the expressions to compute the optimal taxes or subsidies in the Appendix, we have expressed all relevant variables as a ratio in human capital. As a result, we do not need to set the value for \(B\).

Based on these parameter values, the threshold value of the altruism factor is \(\bar{\beta} = 0.6404\). Thus, the bequest motive is operative if \(\beta > 0.6404\) and inoperative if \(\beta < 0.6404\). Then, depending on the case of \(\beta > 0.6404\) or \(\beta < 0.6404\), we simulate the optimal tax policy. These results are as follows.

First, under operative bequest motive \((\beta > 0.6404)\), the optimal tax policy is in Figure 2. Panel (a) is the optimal education tax rate, which is decreasing with the social weight \(\lambda\) but increasing with the
The results indicate that education is subsidized when the altruism factor is smaller than the social weight ($\beta < \lambda$), but is taxed if otherwise. Panel (b) is the optimal tax rate to bequests. The estate tax rate is also decreasing with the social weight and increasing with the altruism factor. The bequest is subsidized when the altruism factor is smaller than the social weight ($\beta < \lambda$), but is taxed if otherwise. Finally, to balance the government budget, the working-aged are taxed a lump sum when the altruism factor is smaller than the social weight ($\beta < \lambda$), but are subsidized a lump sum if otherwise. Panel (c) is these lump-sum taxes or subsidies as a fraction of wage income, $\frac{\tau_{w}}{w}$. 

Next, under inoperative bequest motive ($\beta < 0.6404$), the optimal taxes or subsidies are reported in Figure 3. Panel (a) is a negative education tax rate, and thus an education subsidy. Moreover, there is a government transfer to the old in the form of a public pension. Panel (b) is the replacement ratio, namely, the amount of public pension as a fraction of the wage income, $\frac{\tau_{p}}{w} > 0$. Finally, to balance the government budget, the expenditures on education subsidies and pensions are financed by a lump-sum tax in the period of work. Panel (c) is the lump-sum tax as a fraction of wage income, with the tax decreasing in both the social weight and the altruism factor, which is different from Docquier et al. (2007).

Table 2 gives the optimal taxes or subsidies for selective values of $\beta$ and $\lambda$. The panel with $\beta = 0.75$ is under operative bequest motives ($\beta > 0.6404$). Suppose that the social weight is $\lambda = 0.85$, larger than $\beta = 0.75$. Then, education is subsidized at the rate 11.76\% and bequests are subsidized at the rate 13.33\%, with these subsidies financed by taxing the working age a lump sum equal to 5.62\% of the wage income. When the social weight is lower but still higher than the altruism factor, say $\lambda = 0.8$, the subsidy to education and bequests is smaller and as a result, the lump-sum tax to the working age is smaller. However, when the social weight is smaller than the altruism factor, say $\lambda = 0.7$, then education is instead taxed at the rate 7.14\% and bequests are taxed at the rate 6.67\%, with these tax revenues transferring to the working age as a lump sum that is 2.08\% of the wage income. When the social weight is lower further, education and bequests both are taxed more heavily, with a larger lump-sum transfer to the working age as a share of the wage income.$^{22}$

The remaining panels of Table 2 are under inoperative bequest motives ($\beta < 0.6404$). Under

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$^{22}$ Under operative bequest motives, $b^{\eta} > 0$, which, given the threshold $\beta = 0.6404$, requires that $\lambda$ be larger than 0.6404. Alternatively, under inoperative bequest motives, $b^{\eta} < 0$, which, given the threshold $\beta = 0.6404$, requires that $\lambda$ be smaller than 0.6404.
inoperative bequest motives, the social weight is smaller than the threshold of the altruism factor and larger than the actual altruism factor. Suppose that the altruism factor is $\beta=0.5$. When the social weight is $\lambda=0.6$, larger than $\beta=0.5$, then education is subsidized at the rate 16.67% and the old is subsidized a lump sum equal to 3.96% of the wage income, with these subsidies financed by taxing the working age a lump sum equal to 7.1% of the wage income. Suppose that the social weight $\lambda$ is lower to $\lambda=0.55<\beta=0.5$. Then, the subsidy to the old increases, but the subsidy to education decreases, with a zero education subsidy when the social weight is equal to the altruism factor. Alternatively, suppose that the altruism factor is lower to $\beta=0.4$. Then, when the social weight is $\lambda=0.6$, education is subsidized at the rate 33.33% that is higher than that when $\beta=0.5$, but the old is subsidized a lump sum equal to 3.96% of the wage income equal to that when $\beta=0.5$. With the same transfer to the old but a larger subsidy to education, the working age is taxed a lump sum equal to 10.60% of the wage income that is larger than that when $\beta=0.5$.

A number of studies have estimated the degree of intergenerational altruism in a Barro (1974)-type model. Assuming a cohort of 25 years, Han and Mulligan (2001) estimated the altruism factor at $\beta=0.295$ and Barczyk (2016) attained $\beta=0.280$. Assuming a cohort of 30 years, Nishiyama (2002) obtained $\beta=0.512$, while Bellettini et al. (2017) found the value of $\beta$ in the range of 0.228 and 0.279. All these values are smaller than the threshold value $\beta=0.640$ and thus, inoperative bequest motives are more in line with the data. As such, the proper policy is to offer subsidies to education investment and provide public pension benefits to the old financed by taxing the working age a lump sum.

5. Concluding Remarks

This paper studies suboptimality of physical and human capital accumulation in a three-period OLG model with altruism. In our model, parents may have incentives to finance education and leave bequests for children. We find that parents always finance education for children, but they may or may not leave bequests for children. The extent of altruism plays an important role in determining the suboptimality of physical and human capital accumulation. As a result, the extent of altruism governs whether the policy of public education financing and public pensions can help restore the optimality.

We find that, under a large extent of altruism and thus operative bequest motives, there are two regimes of suboptimality. In this case, both human and physical capital under-accumulate if the altruism factor is smaller than the social weight, but both over-accumulate if otherwise. By contrast, under a smaller extent of altruism and thus inoperative bequest motives, there are three regimes of suboptimality. In this case, both human and physical capital under-accumulate when the social weight is large, both over-accumulate when the social weight is small, and human capital under-accumulates while physical capital over-accumulates when the social weight is in between. These results are very different from the
existing strand of work.

Regarding the policy to restore optimality, we envisage how the extent of altruism affects whether public education financing and public pensions are necessary. When the bequest motive is operative, it is optimal to subsidize both education and bequests, financed by lump-sum taxes of the working age, if the altruism factor is smaller than the social weight, but to tax both education and bequests with the tax revenue transferring to the working age if otherwise. In this case, a public pension program does not need, and public education financing may or may not need in order to restore the optimality. By contrast, when the bequest motive is inoperative, it is always optimal to subsidize education, financed by taxing the working age a lump sum. Moreover, the bequest is zero in equilibrium, but the socially optimal bequest is negative, indicating that the retired consume less than the optimum. Thus, it is optimal to offer a lump-sum benefit to the retired, financed again by taxing the working age a lump sum, an intergenerational program like public pensions. In this case, both public education financing and public pensions are necessary to help restore the optimality.

Finally, we numerically illustrate the optimal tax policy. Our quantitative exercise suggests that inoperative bequest motives are more in line with the data. Therefore, public education financing and public pensions are necessary policies.

Appendix

In this Appendix, first we derive the equation that calculates the threshold value of the altruism factor in the competitive equilibrium. Then, we derive the equations that are used to compute the optimal tax and subsidy as functions of the altruism factor and the social weight.

A. Calculation of the threshold value of the altruism factor

To derive the expression for calculating the threshold value of the altruism factor, we will express all relevant variables in the competitive equilibrium as functions of the altruism factor.

First, based on the parametric forms of the utility and technology in (26a)-(26c), we rewrite the capital to human capital ratio in the competitive equilibrium \( \hat{k}^{CE} \) in (12a) as a function of \( \beta \):

\[
\hat{k}^{CE} = \left( \frac{\beta A}{n} \right)^{\frac{1}{\gamma - a}} \equiv \hat{k}^{CE}(\beta) \tag{A1}
\]

Next, note the result \( \phi'(\hat{\epsilon}^{CE}) = \chi \frac{n}{\gamma \hat{\epsilon}^{CE}} = \chi \frac{\hat{\epsilon}^{CE}}{k^{CE}} \), where \( \hat{\epsilon}^{CE} \equiv \frac{\hat{e}^{CE}}{k^{CE}} \) denotes the education expenditure to human capital ratio in the competitive equilibrium. Then, using the optimal condition (12b), we can express the education expenditure to human capital ratio as a function of \( \beta \) as follows.
Moreover, let $c^{CE}$ be the consumption of the working age to human capital ratio and $d^{CE}$ be the consumption of the retired to human capital ratio in the competitive equilibrium. If we denote their sum by $C^{CE} \equiv c^{CE} + d^{CE}$, we can use (14), (12a) and (A2) to derive the sum as a function of $\beta$ as follows.

$$C^{CE} \equiv \left( \frac{1}{\beta} - \left( \frac{1-\alpha}{\alpha} \chi - 1 \right) n \tilde{k}^{CE} (\beta) \right) \equiv \tilde{C}^{CE} (\beta).$$ \hfill (A3)

Then, using (12a) in the steady state, (8a) gives $\tilde{m}^{CE} = n$. It is clear that (A3) enables us to express $c^{CE}$ and $d^{CE}$ as a function of $C^{CE}$ and thus, a function of $\beta$ as follows.

$$c^{CE} = \frac{\beta}{\rho + \beta} \tilde{C}^{CE} (\beta),$$ \hfill (A4a)

$$d^{CE} = \frac{\rho n}{\rho + \beta} \tilde{C}^{CE} (\beta).$$ \hfill (A4b)

Further, we denote the savings to human capital ratio in the competitive equilibrium in the steady state by $S^{CE} \equiv \frac{\tilde{s}^{CE}}{\tilde{k}^{CE}}$. We can use (7) and (A1) to express the ratio as a function of $\beta$ as follows.

$$S^{CE} = n \tilde{k}^{CE} (\beta).$$ \hfill (A5)

Finally, using (A4b) and (A5), (4b) gives the bequest to human capital ratio in the competitive equilibrium in the steady state, denoted by $b^{CE} \equiv \frac{\tilde{b}^{CE}}{\tilde{k}^{CE}}$. With (A1), the ratio is a function of $\beta$ as follows.

$$b^{CE} = \frac{R S^{CE} - \tilde{d}^{CE}}{n} = \left[ \frac{1}{\beta} - \frac{\rho}{\rho + \beta} \left( \frac{1}{\beta \alpha} - \frac{1-\alpha}{\alpha} \chi - 1 \right) \right] n \tilde{k}^{CE} (\beta).$$ \hfill (A6)

When $b^{CE} (\beta) = 0$, we will obtain the threshold value of the altruism factor $\bar{\beta}$.

B. Calculation of the optimal tax policy as a function of altruism factor and social weight.

To derive these equations, first, based on the parametric forms in (26a)-(26c), we rewrite the capital to human capital ratio in the social optimum $\tilde{k}^{SP}$ in (18a) as a function of $\lambda$:

$$\tilde{k}^{SP} = \left( \frac{\lambda A}{n} \right)^{\frac{1}{\gamma-a}} \equiv \tilde{k}^{SP} (\lambda).$$ \hfill (B1)

Next, we denote the education expenditure to human capital ratio in the social optimum by $e^{SP} \equiv \frac{\tilde{e}^{SP}}{\tilde{k}^{SP}}$. The optimal condition (18b), along with (B1), gives the ratio as a function of $\lambda$ as follows.
In addition, let $\bar{c}_{SP}$ be the consumption of the working age to human capital ratio and $\bar{d}_{SP}$ be the consumption of the retired to human capital ratio in the social optimum. If we denote their sum by $\bar{c}_{SP} \equiv \bar{c}_{SP} + \frac{\bar{d}_{SP}}{\pi}$, then we can use (14), (18a), (B1) and (B2) to write this ratio as a function of $\lambda$ as follows.

$$\bar{C}_{SP} = A\left(\bar{k}_{SP}\right)^\alpha - n\left(\bar{c}_{SP} + \bar{k}_{SP}\right) = \left(\frac{1}{\lambda \alpha} - \frac{1-\alpha}{\alpha} \chi - 1\right)n\bar{k}_{SP}(\lambda) \equiv \bar{C}_{SP}(\lambda). \quad (B3)$$

Moreover, using (17a) in the steady state, (18a) gives $\frac{\bar{d}_{SP}}{\bar{c}_{SP}} = \frac{\rho}{\pi}$. Then, (B3) enables us to express $\bar{c}_{SP}$ and $\bar{d}_{SP}$ as a function of $\bar{C}_{SP}$ and thus, a function of $\lambda$ as follows.

$$\bar{c}_{SP} = \frac{\lambda}{\rho + \lambda} \bar{C}_{SP}(\lambda), \quad (B4a)$$

$$\bar{d}_{SP} = \frac{\rho n}{\rho + \lambda} \bar{C}_{SP}(\lambda). \quad (B4b)$$

Furthermore, denote by $\bar{s}_{SP} \equiv \frac{\bar{s}_{SP}}{\bar{k}_{SP}}$ the savings to human capital ratio in the social optimum. Then, using (B1) and the steady-state version of (7), we can express this ratio as a function of $\lambda$.

$$\bar{s}_{SP} = nk\bar{k}_{SP}(\lambda). \quad (B5)$$

We are ready to derive the optimal tax policy. Notice that when $\beta$ is larger than the threshold $\bar{\beta}$, the bequest motive is operative. In this case, according to Proposition 4, the education tax rate is $\bar{\tau}_e^t = \frac{\beta}{\tau} - 1$ and the estate tax rate is $\bar{\tau}_e^b = 1 - \frac{1}{\beta}$, which are a function of $\beta$ and $\lambda$. In this case, there is a lump-sum tax in the period of the working age. On the other hand, when $\beta$ is smaller than $\bar{\beta}$, the bequest motive is inoperative. According to Proposition 5, the education tax is $\bar{\tau}_e^t = \frac{\lambda}{\tau} - 1$, which is a function of $\beta$ and $\lambda$. In this case, there is a pension in the period of retirement and a lump-sum tax in the period of the working age. Below, we derive these lump-sum taxes and pensions as functions of $\beta$ and $\lambda$.

For the retired, the lump-sum tax is zero $\bar{T}^0 = 0$ when the bequest motive is operative and is negative $\bar{T}^0 = nb_{SP} < 0$, when bequest motive is inoperative. Denote by $\bar{h}_{SP} \equiv \frac{\bar{h}_{SP}}{\bar{k}_{SP}}$ the lump-sum tax of the retired to human capital ratio. Then, if we use (18a), (22), (B3), (B4a), (B4b) and (B5), under an inoperative bequest motive, we can obtain the ratio $\bar{T}^0$ as a function of $\lambda$ as follows.

$$\bar{T}^0 = \frac{nb_{SP}}{h_{SP}} = f'(\bar{k}_{SP})nh_{SP} - \frac{d_{SP}}{h} = \left[\alpha - \frac{\rho}{\rho + \lambda} + \frac{\rho \lambda}{\rho + \lambda}((1-\alpha)\chi + \alpha)\right]n^{\frac{1-\alpha}{\alpha}}(\lambda \alpha)^{\frac{\alpha}{1-\alpha}}(A)^{\frac{1}{1-\alpha}}. \quad (B6a)$$

Following Docquier et al. (2007), we express the amount of pensions to the retired as a fraction of
the wage income, $\bar{T}^{\sigma} \equiv \frac{e^{\sigma}}{w^{\sigma}}$. Then, using the fact that $w = (1 - \alpha) f'(\tilde{k}^{SP})$ and (17a) in the steady state, this replacement ratio is as follows.

$$\bar{T}^{\sigma} = \frac{n}{1 - \alpha} \left[ \alpha - \frac{\rho}{\rho + \lambda} + \frac{\rho \lambda}{\rho + \lambda} \left( (1 - \alpha) \chi + \alpha \right) \right] \equiv \bar{T}^{\sigma}(\lambda).$$ (B6b)

Likewise, denote by $\tilde{T}^{m} \equiv \frac{\tilde{r}}{h}$ the ratio of the lump-sum tax in the period of work to human capital. When bequest motive is operative, $\tilde{T}^{m} = -\tilde{r}^{s} n e^{-x} - \tilde{r}^{b} b_{t}$. Then, using (19a), (B2), (B3), (B4a), (B4b) and (B5), we express the ratio of the lump-sum tax in the period of work to human capital as a function of $\beta$ and $\lambda$ as follows.

$$\tilde{T}^{m} = (\lambda - \beta) \left\{ (1 - \alpha) \chi + \frac{1}{\beta} \left[ \alpha - \frac{\rho}{\rho + \lambda} + \frac{\rho \lambda}{\rho + \lambda} \left( (1 - \alpha) \chi + \alpha \right) \right] \right\} \equiv \tilde{T}^{m}(\lambda, \beta).$$ (B7a)

If we express the lump-sum tax in the period of work as a fraction of the wage income, $\tilde{T}^{m} \equiv \frac{\tilde{r}^{x}}{w^{\sigma}}$, using the fact that $w = (1 - \alpha) f'(\tilde{k}^{SP})$ and (17a) in the steady state, this ratio is as follows.

$$\tilde{T}^{m} = \frac{\lambda - \beta}{1 - \alpha} \left\{ (1 - \alpha) \chi + \frac{1}{\beta} \left[ \alpha - \frac{\rho}{\rho + \lambda} + \frac{\rho \lambda}{\rho + \lambda} \left( (1 - \alpha) \chi + \alpha \right) \right] \right\} \equiv \tilde{T}^{m}(\beta, \lambda).$$ (B7b)

Alternatively, when bequest motive is inoperative, $\tilde{T}^{n} = -n \tilde{r}^{c} - \tilde{b}^{SP}$. Then, using (19a), (B2), (B3), (B4a), (B4b) and (B5), we express the lump-sum tax in the period of work to human capital $\tilde{T}^{n} \equiv \frac{\tilde{r}^{n}}{h}$ as a function of $\beta$ and $\lambda$ as follows.

$$\tilde{T}^{n} = \left\{ (\lambda - \beta)(1 - \alpha) \chi - \left[ \alpha - \frac{\rho}{\rho + \lambda} + \frac{\rho \lambda}{\rho + \lambda} \left( (1 - \alpha) \chi + \alpha \right) \right] \right\} \equiv \tilde{T}^{n}(\beta, \lambda).$$ (B8a)

Again, by expressing the lump-sum tax in the period of work as a fraction of the wage income, $\tilde{T}^{n} \equiv \frac{\tilde{r}^{n}}{w^{\sigma}}$, this ratio is a function of $\beta$ and $\lambda$ as follows.

$$\tilde{T}^{n} = \frac{1}{1 - \alpha} \left\{ (\lambda - \beta)(1 - \alpha) \chi - \left[ \alpha - \frac{\rho}{\rho + \lambda} + \frac{\rho \lambda}{\rho + \lambda} \left( (1 - \alpha) \chi + \alpha \right) \right] \right\} \equiv \tilde{T}^{n}(\beta, \lambda).$$ (B8b)

References


Lucas, R.E., Jr. (1988) On the mechanics of economic development. *Journal of Monetary Economics* 22, 3-


Table 1 Optimal Policy

<table>
<thead>
<tr>
<th>Bequest motives</th>
<th>Taxes or Subsidies</th>
</tr>
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<tr>
<td>Operative $\beta &gt; \beta_0$</td>
<td>Regime II ($\lambda &lt; \beta$)</td>
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<td>$\hat{\tau}_i^e &gt; 0$, $\hat{\tau}_i^b &gt; 0$</td>
<td>$\hat{\tau}_i^e &lt; 0$, $\hat{\tau}_i^b &lt; 0$</td>
</tr>
<tr>
<td>$\hat{T}_i^a = 0$, $\hat{T}_i^m = \hat{T}_i^{m1} + \hat{T}_i^{m2} &lt; 0$</td>
<td>$\hat{T}_i^a = 0$, $\hat{T}_i^m = \hat{T}_i^{m1} + \hat{T}_i^{m2} &gt; 0$</td>
</tr>
<tr>
<td>$\hat{T}_i^{m1} = -\hat{\tau}_i^e n\hat{e}_i &lt; 0$, $\hat{T}_i^{m2} = -\hat{\tau}_i^b \hat{b}_i &lt; 0$</td>
<td>$\hat{T}_i^{m1} = -\hat{\tau}_i^e n\hat{e}_i &gt; 0$, $\hat{T}_i^{m2} = -\hat{\tau}_i^b \hat{b}_i &gt; 0$</td>
</tr>
<tr>
<td>Inoperative $\beta &lt; \beta_0$</td>
<td>Regime V ($\lambda &lt; \lambda_2 + \beta$)</td>
</tr>
<tr>
<td>$\hat{\tau}_i^e &lt; 0$</td>
<td>$\hat{T}_i^o = nb_i^{sp} &lt; 0$, $\hat{T}_i^a = \hat{T}_i^{m3} + \hat{T}_i^{m4} &gt; 0$</td>
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Note: $\hat{\tau}_i^e$: education taxes; $\hat{\tau}_i^b$: bequest tax; $\hat{T}_i^{o}$: Lump-sum tax to the retired; $\hat{T}_i^{m3}, \hat{T}_i^{m4}, \hat{T}_i^{m1}, \hat{T}_i^{m2}$ and $\hat{T}_i^{m4}$: Lump-sum tax to working agents.

Table 2. Numerical Optimal Policy

<table>
<thead>
<tr>
<th>$\beta$</th>
<th>$\lambda$</th>
<th>$\tau_i^e$</th>
<th>$\tau_i^b$</th>
<th>$\hat{T}_i^o / (w_h)$</th>
<th>$\hat{T}_i^m / (w_h)$</th>
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Note: The threshold is $\beta = 0.6404$. Under inoperative bequest motives, $b_i^{sp} < 0$ which in turn requires that $\lambda$ be smaller than 0.6404. By contrast, under operative bequest motives, $b_i^{sp} > 0$, which in turn requires that $\lambda$ be larger than 0.6404.
Figure 1: Regimes of physical and human capital accumulation
Figure 2: Numerical results under operative bequest motives \((\beta > 0.6404)\)

(a) Education tax/subsidy \((\bar{\tau}_e = \frac{p}{\beta} - 1)\); (b) estate tax/subsidy \((\bar{\tau}_b = 1 - \frac{p}{\beta})\); and (c) Lump-sum tax/subsidy as a fraction of wage in periods of work \((\frac{r}{w_0})\).
Numerical results under inoperative bequest motives ($\beta < 0.6404$)

(a) Education tax/subsidy ($\bar{\tau}^e = \frac{\beta}{\lambda^2} - 1$); (b) replacement ratio to the retired ($-T^w \equiv \frac{\nu wT}{\nu wT} > 0$); and (c) lump-sum tax/subsidy as a fraction of wage in periods of work ($T^m \equiv \frac{\bar{\tau}^s}{\nu wT} > 0$).