A pay-as-you-go pension system in a two-sector model

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Abstract

In this study, we investigate how demographic shrinking affects the effectiveness of a pay-as-you-go pension system by using a two-sector overlapping generations model. The seminal paper, Fanti and Gori (2012) showed that demographic shrinking does not necessarily cause a decrease in pension benefit. We revisit the existing results by extending the original one-sector model into a two-sector model. We find that there is a positive effect on the pension benefit of the shrinking population as well and derive relevant conditions in the two-sector model. We also present numerical simulations. Our results suggest that the positive effect of population decline on the pension benefit is limited.

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1 Introduction

In this paper, we investigate how demographic shrinking affects the effectiveness of pay-as-you-go (henceforth PAYG) pension system by using a two-sector overlapping generations (henceforth OLG) model. The pension system is maintained by under the PAYG pension system and the continuous decline in the fertility rate can be seen in many developed countries. According to the World Bank’s (2018) World Development Indicators, the total fertility rate between 1968 and 2016 fell from 2.13 to 1.44 in Japan, from 2.64 to 1.8 in the U.S., from 3.65 to 1.21 in Hong Kong, from 2.38 to 1.5 in Germany, and from 2.69 to 1.96 in France. This continuous decline, in turn, raises great concern about the sustainability of the pension system. It seems to be obvious that the decline in the fertility rate lowers the pension benefit received by the older generation under PAYG pension system because the lower fertility rate suggests lower pension contribution.

However, Fanti and Gori (2012) found that taking a general equilibrium effect into account, the decrease in the population growth rate can increase the pension benefit because of the wage increase. In their one sector model, a decrease in the population growth rate increases per capita capital, which leads to an increase in the wage rate. The increases in the wage rate can compensate a reduction of pension contribution caused by the decreased number of the young who pay the pension contribution.

We revisit the Fanti and Gori (2012) result that demographic shrinking does not necessarily decrease PAYG pension benefit per capita by considering a two-sector OLG model. We construct a two-sector growth model with an OLG demographic structure. Under this setting, we examine the effects of demographic change on the pension benefit. It will be shown that as in Fanti and Gori (2012), there are two effects of demographic change on the pension: a direct effect, which reduces the pension benefit as the population growth rate declines and an indirect effect, which increase the pension benefit as the population growth rate decreases as long as the equilibrium is stable. It is well known that one of the necessary conditions for saddle stability in the two-sector neoclassical OLG model is the investment goods sector is more capital-intensive than the consumption goods sector (see Golor, 1992 and Cremers, 2006). This assumption ensures the sign of the Rybczinski and Stolper-Samuelson effects and then the sign of the indirect effect.
Our paper confirms the result of Fanti and Gori (2012) that demographic change has two effects on the pension benefit in a general equilibrium model. However, the mechanism behind the results is somewhat different. In Fanti and Gori (2012), the indirect effect comes from an increase in the wage rate caused by the increase in the capital stock per capita. In our model, the increase in the capital stock increase the investment (capital-intensive) goods production (the Rybczinski effect). To satisfy the market equilibrium condition for the investment goods, the saving must increase. With the assumed capital intensity and the Stolper-Samuelson effect, a decrease in $p$ raises the wage rate and the saving. The increase in the wage rate makes the indirect effect positive.

The remainder of this paper is organized as follows. Section 2 describes the two-sector OLG model. Section 3 presents the main results. Section 4 conducts several numerical analyses. Section 5 presents the concluding remarks.

2 The Model

This section presents an OLG model incorporating a pay-as-you-go pension system into the neoclassical two-sector growth framework.

2.1 Firms

There are two sectors in this economy, consumption goods sector (sector $C$) and investment goods sector (sector $I$) and there exist an infinite number of homogeneous firms in both sectors. These firms compete in perfectly competitive markets and maximize their profits subject to the production function. Firms in both sectors employ labor and capital and have the same constant-returns-to-scale technology. For sector $i (i = C$ and $I$), let $f_i$ denote the technology when expressed in intensive form. The $f_i$ are assumed to be twice continuously differentiable and satisfy, $f(0) = 0$, $f_i' > 0$, $f''_i < 0$, $f_i''(0) = \infty$, $f_i'(\infty) = 0$, where $f_i' = df_i/dk_i$, $f_i'' = df_i'/dk_i$ and $k_i$ is the ratio of physical capital to labor in the $i$th sector. The production function of a firm in both sectors is represented by

\begin{align}
Y_{Ct} &= f_C(k_{Ct})L_{Ct}, \\
Y_{It} &= f_I(k_{It})L_{It},
\end{align}

where $L_i, (i = C, I)$ is labor employed in each sector respectively and subscript $t$ is time index. Labor’s and capital’s resource constraints that the
The economy confronts in period $t$ are

$$l_C + l_I = 1,$$

$$l_Ck_C + l_Ik_I = k_t,$$

where $l_t = L_t/L$. $K_t$ is total capital stock and $L_t$ is total labor in period $t$. From (3) and (4), the labor ratios of each sector are

$$l_C = \frac{k_t - k_{It}}{k_C - k_{It}}, l_I = \frac{k_t - k_{Ct}}{k_I - k_{Ct}}.$$  

Letting the consumption goods numeraire and the price of the investment goods $p$, the first order conditions of firms are as follows:

$$w_t = f_C - k_tC'f_C' = p_t (f_C - k_{It}f_I'),$$

$$r_t = f_C' = p_t f_I',$$

where $w_t$ and $r_t$ are the wage rate and the rental paid on capital services in period $t$, respectively. As is well known in the two-sector model, from (6) and (7) the capital-labor ratio in each sector, the interest rate and the wage rate can be written as functions of the price of the investment goods, $p$, as follows:

$$k_C = \tilde{k}_C(p_t), \quad k_I = \tilde{k}_C(p_t),$$

$$\frac{dk_C}{dp} = \frac{f_I}{f'_C(k_C - k_t)},$$

$$\frac{dk_I}{dp} = \frac{f_C}{p^2 f''_C(k_I - k_t)},$$

$$r_t = r(p_t), \quad \frac{dr}{dp} = \frac{f_I}{k_I - k_C},$$

$$w_t = w(p_t), \quad \frac{dw}{dp} = \frac{-k_C f_I}{k_I - k_C}.$$  

It is natural that we assume that the investment goods sector is more capital intensive than the consumption goods sector, that is, $\tilde{k}_I(p) > \tilde{k}_C(p)$ for all $p$. Then, the Stolper-Samuelson effects are established as follows:

$$w' \equiv \frac{dw(p)}{dp} < 0, \quad r' \equiv \frac{dr(p)}{dr} > 0.$$  

Combining (5), (8), (9), (10), and (11) with the production functions, we can obtain the production level of each sector as a function of $p$ and $k$: $x_C = \tilde{x}_C(p_t, k_t) = l_C f_C(k_C) = \frac{k_t - \tilde{k}_t(p_t)}{k_C(p_t) - \tilde{k}_t(p_t)} f_C(\tilde{k}_C(p_t)),$

$$x_I = \tilde{x}_I(p_t, k_t) = l_I f_I(k_I) = \frac{k_t - \tilde{k}_C(p_t)}{k_I(p_t) - \tilde{k}_C(p_t)} f_I(\tilde{k}_C(p_t)).$$
The following property of the above functions are also obtained:\(^1\)

\[
\tilde{x}_{tp} \equiv \frac{\partial \tilde{x}_t}{\partial p} > 0, \quad \tilde{x}_{tk} \equiv \frac{\partial \tilde{x}_t}{\partial k} = \frac{1}{k_{1t} - k_{Ct}}f_l(k_{1t}(p_t)) > 0. \tag{15}
\]

### 2.2 Individuals

Each individual lives two periods, the young period and the old period. Population grows at the rate of \(n\) exogenously. We assume that the individuals work in the young period only so that \(L_t\) denote the number of the young in period \(t\) as well as the total labor in period \(t\). Population grows exogenously, at the gross population growth rate, \(n > 1\), that is, \(L_{t+1} = nL_t\).

The individuals obtain utility from consumption in both the young and the old periods. The utility function of the young in period \(t\) is given in log-linear form as follows:

\[
\log c_{1t} + \beta \log c_{2t+1}, \tag{16}
\]

where \(c_{1t}\) and \(c_{2t+1}\) are the level of consumption in the young period and in the old period in period \(t\), respectively, and \(\beta\) is the time discount factor.

By working in the young period, the individuals earn the wages for consumption, savings, and childrearing. Childrearing is costly and requires \(q\) units of the wage to bring up one child. The individuals also pay income tax for pension contribution. The old individuals retire from work and consume using their savings and pension benefit. The budget constraints of the individuals in each period are

\[
c_{1t} + s_t + qw_t n = w_t (1 - \theta), \tag{17}
\]

\[
c_{2t+1} = r_{t+1} \frac{s_t}{p_t} + \delta_{t+1}, \tag{18}
\]

where \(q\) is a child-rearing cost, \(\theta\) the income tax rate, \(\delta\) the pension benefit. To make sure that the individuals receive positive income in the young period, we assume that \(1 - \theta - qn > 0\).

The utility maximization yields the consumption function for each period and the saving function as follows.

\[
c_{1t} = \frac{1}{1 + \beta} \left[ (1 - \theta - qn) w_t + \frac{\delta_{t+1}}{\rho_{t+1}} \right], \tag{19}
\]

\[
c_{2t+1} = \frac{\beta \rho_{t+1}}{1 + \beta} \left[ (1 - \theta - qn) w_t + \frac{\delta_{t+1}}{\rho_{t+1}} \right], \tag{20}
\]

\[
s_t = \frac{\beta}{1 + \beta} (1 - \theta - qn) w_t - \frac{1}{1 + \beta} \frac{\delta_{t+1}}{\rho_{t+1}}. \tag{21}
\]

\(^1\)See Appendix B for deriving the sign of \(\tilde{x}_{tp}\).
where \( \rho_{t+1} \equiv r_{t+1}/p_t \) stands for the rate of return on capital owned from periods \( t \) to \( t+1 \).

### 2.3 General Equilibrium

In a two-sector model, a market for investment goods is separate from the market for financial capital. In this market, the supply of investment goods should be equal to savings which constitute the demand for investment goods in equilibrium.

\[
\tilde{x}_I(p_t, k_t) = \frac{s_t}{p_t}. \tag{22}
\]

The government maintains a PAYG pension system and finances the pension benefit by taxing labor income. Thus, the budget constraint of the government is as follows:

\[
\delta_t = n \theta w_t. \tag{23}
\]

By incorporating (11), (21), and (23) we can rewrite as follows:

\[
\tilde{x}_I(p_t, k_t) = \frac{\beta}{1+\beta} (1 - \theta - qn) \frac{w(p_t)}{p_t} - \frac{1}{1+\beta} \frac{n\theta w(p_{t+1})}{r(p_{t+1})}. \tag{24}
\]

Assuming that capital depreciates completely after one period, the law of capital accumulation or the market equilibrium condition for financial capital can be described as follows:

\[
K_{t+1} = f_I(k_{t+1})L_{t+1}. \tag{25}
\]

Dividing the both sides by \( L_t \) and using (14), the above equation can be rewritten as follows:

\[
k_{t+1} = \frac{1}{n} \tilde{x}_I(p_t, k_t). \tag{26}
\]

(24) and (26) consist an autonomous difference equation system with respect to \( p_t \) and \( k_t \).

The steady state equilibrium is a situation in which all prices and quantity variables per capita stay constant. Thus, \( k \equiv k_t = k_{t+1} \) and \( p \equiv p_t = p_{t+1} \) hold. At the steady state equilibrium, (24) and (26) yield

\[
\tilde{x}_I(p, k) = \frac{\beta}{1+\beta} (1 - \theta - qn) \frac{w(p)}{p} - \frac{1}{1+\beta} \frac{n\theta w(p)}{r(p)}, \tag{27}
\]

\[
k = \frac{1}{n} \tilde{x}_I(p, k). \tag{28}
\]
These two equations determine the steady state levels of $p$ and $k$. In Appendix A, we examine the Walrasian (static) stability and dynamic stability of the system and find sufficient conditions to ensure the Walrasian and dynamic stability. We assume that the sufficient conditions are satisfied in the following analysis.

3 The Pension Benefit and the Population Growth

Here we analyze the demographic effect on the pension benefit. As the government distributes all amount of collected income tax as pension benefit, the amount of pension benefit, $\delta$, is equal to $n\theta w$. The effect of demographic change on the pension benefit can be calculated as follows.

$$
\frac{d\delta}{dn} = \theta w(p) + \theta n \frac{dw(p)}{dn} = \theta w(p) + n \theta \frac{dp}{dp} \frac{dw}{dn}.
$$

(29)

The first term is a direct effect on the pension benefit: As the number of young generation lowers, the pension contribution and the pension benefit decreases. The second term is an indirect effect: The demographic change affects the price and the wage rate and thus the pension contribution per capita. To identify the indirect effect, we need a general equilibrium analysis.

Totally differentiating (27) and (28) with respect to $p$, $k$, and $n$, we obtain following:

$$
\left( \ddot{x}_{kp} - \frac{\beta}{1+\beta} \frac{w'p - w}{p^2} (1 - \theta - qn) + \frac{n\theta}{1+\beta} \frac{w'r - wr'}{r^2} \right) \frac{\ddot{x}_{1k}}{\ddot{x}_{kp}} - \frac{\ddot{x}_{1k}}{-\frac{\beta}{1+\beta} \frac{qw}{p} - \frac{1}{1+\beta} \frac{\theta w}{r}})
$$

(30)

Then, we can show $dp/dn$ and $dk/dn$ as follows:

$$
\frac{dp}{dn} = \Delta^{-1} \left[ \left( -\frac{\beta}{1+\beta} \frac{qw}{p} - \frac{1}{1+\beta} \frac{\theta w}{r} \right) (\ddot{x}_{1k} - n) - k\ddot{x}_{1k} \right].
$$

(31)

$\Delta$ is the determinant of the coefficient matrix of the left-hand side of (30) and is calculated as follows:

$$
\Delta = \left[ \ddot{x}_{kp} - \frac{\beta}{1+\beta} \frac{w'p - w}{p^2} (1 - \theta - qn) + \frac{n\theta}{1+\beta} \frac{w'r - wr'}{r^2} \right] (\ddot{x}_{1k} - n) - \ddot{x}_{1k} \ddot{x}_{kp}
$$

$$
= n \left( -\frac{n\theta}{1+\beta} \frac{w'r - wr'}{r^2} \right) \left( \left( 1 - \frac{\ddot{x}_{1k}}{n} \right) (1 - \phi_p) - \phi_h \frac{\ddot{x}_{1p}}{n} \right).
$$

(32)
\( \phi \) derived from (24) is a function of \( p_t \) and \( k_t \), which gives the price level in period \( t+1 \), that is, \( p_{t+1} = \phi(p_t, k_t) \), \( \phi_p = \partial \phi / \partial p_t \), and \( \phi_k = \partial \phi / \partial k_t \). We use (A-4) and (A-5) to derive the second equality. By examining the sign of (32), we can establish the following lemma.

**Lemma 1.** As long as the Walrasian and dynamic stabilities hold, the indirect effect is negative.

**Proof.** Since the inside of the square brackets of the last line of (32) is negative to ensure the dynamic stability (see A-9), we confirm that \( \Delta < 0 \). The inside of the brackets of RHS of (31) is negative.\(^2\) It means that the signs of (31) is positive. That is, an increase in the population growth rate raises the price of capital intensive goods. Taking (12) into consideration, the indirect effect in (29) is proved to be negative in our model. \( \square \)

The reason why the indirect effect (the general equilibrium effect) is negative is as follows: Note that the sign of \( dk/dn \) is negative.\(^3\) Thus, a decrease in the rate of population growth raises the capital per worker in the new steady state and increases the output of the investment goods. In a two-sector growth model, the saving of the young should be equal to the output of the investment. With the assumed capital intensity and the Stolper-Samuelson effect, a decrease in \( p \) raises the wage rate and the saving. The increase in the wage rate makes the indirect effect positive.

Equation (29) can be arranged as follows:

\[
\frac{d\delta}{dn} = \theta w(p) \left( 1 + \eta_{wp} \eta_{pm} \right),
\]

where \( \eta_{wp} \equiv (p/w)(dw/dp) \) and \( \eta_{pm} \equiv (n/p)(dp/dn) \). By further calculation, we confirm that \( d^2 p/dn^2 < 0 \) (see Appendix C). Thus, \( 0 < \eta_{pm} < 1 \). If

\[
\text{In the steady state equilibrium,}
\]

\[ \tilde{x}_t = \frac{k - k_C}{k_t - k_C} f_1(\tilde{k}_t(p)) = nk \]

holds. From (14), we know that \( \tilde{x}_{1t} = \frac{1}{k_t - k_C} f_1(\tilde{k}_t(p)) \) holds. Thus, we confirm that \( \tilde{x}_{1t} > n \) around the steady state. Then, the sign of (31) is determined.

\(^3\)From (30), we obtain the following equation:

\[
\frac{dk}{dn} = \Delta^{-1} \left\{ \left[ x_{1p} - \frac{\beta}{1+\beta} \frac{w'}{p} - w \left( 1 - \theta - qn \right) + \frac{n\theta}{1+\beta} \frac{w' - w'}{r^2} \right] k 
\]

\[ - \tilde{x}_{1p} \left( - \frac{\beta}{1+\beta} \frac{qw}{p} - \frac{1}{1+\beta} \frac{\theta w}{r} \right) \right\} < 0. \]

The sign follows from the sufficient condition for the Walrasian stability.
\[-1 < \eta_{wp}, \text{ the direct effect always dominates the indirect effect. If } -1 > \eta_{wp}, \text{ it is possible that indirect effects dominates the direct effect. The magnitude of } \eta_{wp} \text{ depends on the capital intensity of each sector.}\]

**Proposition 1.** The sign of the total effect depends on the elasticity of the wage to the price. Namely, when the elasticity is greater than -1, the total effect is always positive, which means that a decrease in the population growth always lowers the pension benefit. When the elasticity is less than -1, the total effect is ambiguous.

Indeed, if we specify the production function as the Cobb-Douglas type such as \( f_c(k_{Ct}) = k_{Ct}^a \) and \( f_I(k_{It}) = k_{It}^b \), \( \eta_{wp} = a/(a-b) \). Thus, if \( 2a < b \) (the capital intensities of each sector differ so much), the direct effect is always larger than the indirect effect, and if \( 2a > b \), the sign of (29) is ambiguous.

**Corollary 1.** Suppose that the production functions are specified as the Cobb-Douglas type. When the difference between the capital intensities of each sector is large, the direct effect is always larger than the indirect effect. Otherwise, the total effect of the shrinking population on the pension benefit is ambiguous.

## 4 Numerical simulations

We demonstrated that the direct effect and the indirect effect work in opposite directions on the pension benefit but cannot show which effect outweighs the other analytically when the difference between the capital intensities of each sector is not so large. In this section, we conduct some numerical simulations to see the relative magnitude of the direct effect and the indirect effect in certain structural parameters. We are particularly interested in the effects of the changes in the population growth the difference between the capital intensities and \( n \). We specify the production functions of the firms in both sectors as the Cobb-Douglas type: \( f_c(k_{Ct}) = k_{Ct}^a \) and \( f_I(k_{It}) = k_{It}^b \).

We set \( \beta = 0.3, q = 0.3, b = 0.4 \) and \( \theta = 0.15 \). One period in our model corresponds to 30 years.

First, we see the effect of the difference between the capital intensities to supplement the corollary. Fig.1 plots the relationship between the capital intensities and the total effect on the pension benefit. We set \( n = 1.05 \) and change \( a \) from 0.34 to 0.39.\(^4\) Though the value of the total effect is small, it takes a positive value \( a \) from 0.35 to 0.39.\(^5\)

\(^4\)The replacement fertility rate per woman in several industrialized countries is 2.1. In our single-parent model, it corresponds to 1.05.

\(^5\)With \( a = 0.39 \), the total effect is \( 1.49619 \times 10^{-20} \).
The total effect

Figure 1: The relationship between the capital intensity of the consumption goods sector, $a$, and the total effect

Next, we numerically simulate the effect of a change in the population growth rate on the total effect. In Fanti and Gori (2012), there is a threshold value of the population growth rate at which the indirect effect outweighs the direct effect. In our model, the total effect is positive for all the levels of $n$ from 0.95 to 1.5.
5 Concluding Remarks

In this paper, we investigate how demographic shrinking affects the effectiveness of a PAYG pension system by using a two-sector OLG model. We confirm the result of Fanti and Gori (2012) that demographic change on a pension has two effects in a general equilibrium model. We also find that the magnitude of the indirect effect depends on the difference between the capital intensities and diminishes as the difference becomes small.

We conclude this paper by suggesting directions for further research. First, we can analyze the same issue in a different production structure. Sen (2006) pointed out that under the Leontief-type production structure, the stability conditions are ensured when a consumption goods sector is more capital intensive than an investment goods sector. As we revealed that the sign of the indirect effect depends on the capital intensity assumption, the direct effect and the indirect effect may work in the same direction under the Leontief-type production structure. Second, it would be interesting to investigate the relationship between the population growth rate and the pension benefit in the endogenous growth model. As well as the two effects already mentioned in the paper, there should exist another effect through a change in economic growth rate. The third effect may complicate the analysis, but we would also find a more fruitful relationship between the population growth rate and the pension contribution than the one that we have documented in Section 3. Future research should be directed at addressing these aspects.
Appendix A. The stability of the steady state equilibrium

First, we check the Walrasian stability. (27) yields the function of excess demand of the investment goods, \( D(p) \),

\[
D(p) = \frac{\beta}{1 + \beta}(1 - \theta - qn)\frac{w(p)}{p} - \frac{1}{1 + \beta} \frac{n\theta w(p)}{r(p)} - \tilde{x}_t(p, k). \tag{A-1}
\]

The Walrasian stability requires that the excess demand decreases with \( p \). Differentiating (A-1) with respect to \( p \), we have

\[
\frac{dD}{dp} = \frac{\beta}{1 + \beta}(1 - \theta - qn)\frac{w'p - w}{p^2} - \frac{n\theta}{1 + \beta} \frac{w'r - wr'}{r^2} - \tilde{x}_t. \tag{A-2}
\]

Under the assumption that the investment goods sector is more capital intensive, if \( \theta \) is not so large, the Walrasian stability condition is satisfied. In what follows, we assume that the RHS of (A-2) is negative.

Next, we check the dynamic stability. From (24), the price level in period \( t + 1 \) can be expressed as a function of \( p_t \) and \( k_t \), that is, \( p_{t+1} = \phi(p_t, k_t) \). The partial derivatives of \( \phi \) with respect to \( p \) and \( k \) and the sign of them can be obtained as follows. Totally differentiating (24) with respect to \( p_t, k_t, \) and \( p_{t+1} \), we obtain

\[
\begin{align*}
\tilde{x}_t dp_t + \tilde{x}_t dk_t &= \left[ \frac{\beta}{1 + \beta}(1 - \theta - qn)\frac{w'p - w}{p^2} \right] dp_t \\
&\quad - \left[ \frac{n\theta}{1 + \beta} \frac{w'r - wr'}{r^2} \right] dp_{t+1}.
\end{align*} \tag{A-3}
\]

As \( p \equiv p_t = p_{t+1} \) at the steady state, we derive

\[
\begin{align*}
\phi_k &\equiv \frac{\partial \phi}{\partial k_t} = \tilde{x}_t \Gamma > 0, \tag{A-4} \\
\phi_p &\equiv \frac{\partial \phi}{\partial p} = \left[ \frac{\beta}{1 + \beta}(1 - \theta - qn)\frac{w'p - w}{p^2} + \tilde{x}_t \right] \Gamma, \tag{A-5}
\end{align*}
\]

where \( \Gamma \equiv -\frac{1 + \beta}{n\theta} \frac{r^2}{w'r - wr} > 0 \). Moreover, as we assume that the RHS of (A-2) is negative, we confirm that

\[
\phi_p > 1 \tag{A-6}
\]

holds around the steady state equilibrium.

Taking \( p_{t+1} = \phi(p_t, k_t) \) into account and linearizing the dynamic system around the steady state, we have

\[
\begin{pmatrix} p_{t+1} \\ k_{t+1} \end{pmatrix} = \begin{pmatrix} \phi_p & \phi_k \\ \tilde{x}_t/p & \tilde{x}_t/k \end{pmatrix} \begin{pmatrix} dp_t \\ dk_t \end{pmatrix}. \tag{A-7}
\]
Since \( p \) is jumpable and \( k \) is unjumpable, in order for the system to be saddle-stable, \( \lambda_1 > 1 \) and \( 1 > \lambda_2 > 0 \) should hold, where \( \lambda_1(\lambda_2) \) is defined as the larger (smaller) eigenvalue of the coefficient matrix of (A-7).

As is well known, when the trace of the matrix, \( T \), and the determinant, \( D \), satisfy \( D > 0 \) and \( 1 - T + D < 0 \), then \( \lambda_1 > 1 \) and \( 1 > \lambda_2 > 0 \) hold. Each condition is calculated as follows:

\[
D = \phi_p \frac{\ddot{x}_{Ik}}{n} - \phi_k \frac{\ddot{x}_{Ip}}{n} = \frac{\ddot{x}_{Ik}}{n} \Gamma \left[ -\frac{\beta}{1 + \beta} (1 - \theta - qn) \frac{w'p - w}{p^2} \right] > 0. \tag{A-8}
\]

The second equality is obtained by substituting (A-4) and (A-5) into \( D \) and arranging it. The sign is determined by (12), (15), and the sign of \( \Gamma \).

\[
1 - T + D = \left( 1 - \frac{\ddot{x}_{Ik}}{n} \right) (1 - \phi_p) - \phi_k \frac{\ddot{x}_{Ip}}{n}. \tag{A-9}
\]

Thus, \( 1 - T + D < 0 \) is the sufficient condition for the dynamic stability to be guaranteed. We assume that \( 1 - T + D < 0 \) throughout the study.

**Appendix B. Derivation of the sign of \( \ddot{x}_{Ip} \)**

\[
\frac{\partial \ddot{x}_I}{\partial p} = \frac{-\dddot{k}_C(k_I - k_C) - (k - k_C)(\dddot{k}_C')}{(k_I - k_C)^2} f_I + \frac{k - k_C}{k_I - k_C} f_I' k_I' = \frac{-\dddot{k}_C(k_I - k) - (k - k_C)\dddot{k}_C'}{(k_I - k_C)^2} f_I + l_I f_I' k_I' = \frac{1}{(k_I - k_C)^2} \left\{ \left[ -\dddot{k}_C l_C (k_I - k_C) - (k - k_C) \dddot{k}_C' \right] f_I + l_I (k_I - k_C)^2 f_I' k_I' \right\} = \frac{1}{(k_I - k_C)^2} \left\{ \left[ l_C f'_I \right] - (k - k_C) \dddot{k}_C' \right\} f_I + l_I (k_I - k_C) f_I' \frac{f_C}{p'^2 f_I''} = \frac{1}{(k_I - k_C)^2} \left[ l_C f'_I - \frac{l_I f_C}{p'^2 f_I''} (f_I - k_I f'_I + k_C f'_I') \right] > 0.
\]

**Appendix C. Derivation of the sign of \( \frac{d^2p}{dn^2} \)**

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Differentiating $dp/dn$ with respect to $n$, we obtain

$$\frac{d^2p}{dn^2} = \frac{1}{\Delta^2} \left( \frac{1}{1 + \beta} - \frac{\beta}{1 + \theta} \right) \Delta - \left[ \frac{1}{\Delta} \left( \frac{1}{1 + \beta} \frac{q w}{p} - \frac{1}{1 + \beta} \frac{\theta w}{r} \right) (\bar{x}_{1k} - n) - k \bar{x}_{1k} \right] \left[ -X + \frac{\theta}{1 + \beta} \frac{w' r - w r'}{r^2} (\bar{x}_{1k} - n) \right]
$$

$$= \frac{1}{\Delta^2} \left( \frac{1}{1 + \beta} \frac{q w}{p} - \frac{\beta}{1 + \beta} \frac{\theta w}{r} \right) \bar{x}_{1k} \bar{x}_{1p} + k \bar{x}_{1k} \left[ -X + \frac{\theta}{1 + \beta} \frac{w' r - w r'}{r^2} (\bar{x}_{1k} - n) \right] - \left[ \frac{1}{\Delta} \left( \frac{1}{1 + \beta} \frac{q w}{p} - \frac{1}{1 + \beta} \frac{\theta w}{r} \right) (\bar{x}_{1k} - n) - k \bar{x}_{1k} \right] \left[ \frac{\theta}{1 + \beta} \frac{w' r - w r'}{r^2} (\bar{x}_{1k} - n) \right]<0,$$

where $X \equiv -\beta/(1 + \beta) (1 - \theta - q n) (w' p - w)/p^2 + n \theta/(1 + \beta)(w' r - w r')/r^2 + \bar{x}_{1p} > 0$. The sign is determined by (12), (15) and, the condition $\bar{x}_{1k} - n > 0$ obtained before.

References


