Effective Marginal Tax Rates of Pension Systems

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Abstract

This paper aims to answer the question of how pension systems impact labour supply in a framework where agents face uncertainty about their future incomes. I demonstrate that the effective marginal tax rate, which takes into account both payroll taxes and future benefits, can be derived directly from the first order conditions of an optimizing household, and decompose it into five intuitive components. Using a lifecycle model calibrated to the US economy, I then calculate that the effective marginal tax rate of 4% lies significantly below the statutory rate of 11%. Contrary to previous work, I find that this rate does not differ significantly by income. However, this homogeneity hides significant differences in the composition of the effective tax rates, with borrowing constraints driving the tax rate for low income households, while high income households are primarily impacted by the redistribution inherent in the US pension system. Finally, I show that it would be possible to reduce this rate by an additional 3% points by replacing the current US pension system with a linear pension that pays the same average replacement rate.
1 Introduction

For economists, marginal tax rates are central to understanding labor supply. A large part of the optimal tax literature is concerned with managing the trade-off between insurance and distortions to labor supply through high marginal tax rates. Following Prescott (2004), there is also a sizable literature that uses marginal tax rates in order to understand differences in labor supply patterns across countries.

However, measuring marginal tax rates properly is a more complicated problem than it appears at first glance. The reason for this is that taxes do not simply "vanish into the ether", but rather are used to finance expenditures, some of which are linked to current or past incomes. This implies, for example, a significantly higher effective marginal tax rate at the bottom of the income distribution due to benefit withdrawal, than what would be suggested by considering only the statutory tax rates.

Another area where the statutory rate is likely to be a poor estimate of the true distortion are social security contributions, especially pension taxes, as future benefits are often linked to past contributions, potentially significantly reducing the distortions below the statutory rate. To see this, consider two extreme examples. Both country $A$ and $B$ have a public pension system that levies a 10% payroll tax on earnings to fund this system. In country $A$, pensions are paid as a lump-sum transfer to all pensioners, while in country $B$, pensioners receive a pension that is equal to their contributions plus interest at the market rate. Thus, workers in country $A$ see the statutory rate as a full tax, as their income is taxed at a 10% marginal rate, while pension benefits depend on aggregate labour supply, but do not increase with labour supply. On the other hand, workers in country $B$ do not perceive their pension contributions as a tax at all, as the marginal tax they face is perfectly offset by the marginal increase in pension benefits. This, admittedly oversimplified, example demonstrates the importance of viewing the pension system as a whole when considering its impact on labor supply decisions.

In 2017, the OECD estimated that social security contributions, most of which are earmarked for the pension system, constitute 33% of the labour wedge for a US worker with average wages, and 42% for workers making less than 90% of the average wage. Due to the looming solvency crisis of many pension systems, this fraction is likely to go up in the absence of fundamental reforms. Thus it is of primary importance to understand the specific distortions induced by the pension system and what steps might be taken to reduce them.

This paper demonstrates that the effective marginal tax rates of the US pension system can be directly derived from households’ optimality con-

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ditions in a stochastic lifecycle model with ex-ante heterogeneity and idiosyncratic risk. Furthermore, I show that the effective tax rate can be decomposed into five intuitive components, that relate to i) inter-generational redistribution, ii) intra-generational redistribution, iii) demographic conditions, iv) borrowing constraints, and v) the insurance provided through the pension system. This model is then calibrated to the US economy in order to quantify these net distortions, as well as their decomposition. Over the course of workers’ lives, the effective marginal tax rate drops from an average of 8% at age 21 to −1% at age 63, the average effective rate over the lifecycle being significantly lower than the payroll tax of 11%, at 4%. Despite the very progressive US pension system, which sees replacement rates as high as 90% and as low as 15%, there is very little variation of effective tax rates by income, with the difference between average rates faced by high- and low-income workers staying below 2% points for most of their working lives. Furthermore, this variation is exactly opposite in sign to what one might first expect, with low income workers subject to the largest effective tax rates. However, this relative homogeneity hides significant heterogeneity between income groups that can be uncovered by means of decomposing the effective tax rate into its components. While for low-income workers, borrowing constraints play a sizable role, high-income workers are subject to much higher intra-generational redistribution.

The decomposition of the effective tax rates also allows me to appraise the effects of the inherent progressivity of the pension system on labour supply in more detail. I show that this mainly impacts two of the components of the labour distortion: the intra-generational redistribution, and the insurance component. I demonstrate that the latter is quantitatively small for all agents, while the former is sizable on average. This is due to the fact that workers pay for the average replacement rate through the payroll tax, but take into account their expected marginal replacement rate when making labour supply decisions. Due to the strong progressivity of the current US system, the average marginal replacement rate lies significantly below the average replacement rate, thereby driving up the average effective tax rates.

Motivated by this finding, I consider a reform that replaces the current pension system with a linear pension system that provides the same average replacement rate. I find that such a reform would reduce the average effective marginal tax rate of the US pension system by 3% points, leading to a 1% increase in effective labour supply in the economy.

The remainder of the paper is structured as follows. The next section provides a brief overview of the related literature. In section 3, I demonstrate how the effective marginal tax rate can be derived from the household optimization problem in a standard lifecycle model with a realistic pension system, while section 4 demonstrates how the effective tax rate can be decomposed. Section 5 calibrates this model and discusses the current effective
marginal tax rate of the US pension system quantitatively, while section is concerned with the policy experiment. Finally, section concludes.

2 Literature Review

This paper is not the first to consider the question of the net marginal tax rate associated with the social security payroll tax. Especially in the literature on the privatization of social security, authors have long cited the distortions introduced by the payroll tax as one of the ways in which a privatized system may be superior. Significant contributions to this discussion include Homburg (1990), Breyer and Straub (1993), as well as Feldstein and Samwick (1998). The first two consider social security in a simple two-period OLG economy, while the latter takes into account a more realistic lifecycle.

What is common to this literature is the effective tax rate has been approximated very roughly by the differences in returns between private savings and the public pension system. While I show that this difference does indeed impact the effective tax rate of the pension system, it is not the quantitatively most important component.

Following work by Gordon (1982), Browning (1985), and Burkhauser and Turner (1985), Feldstein and Samwick (1992) were among the first to consider the net effective tax rate of social security by taking into account the specific design of the pension system. Similar to this study, they find that effective tax rates are decreasing over the lifecycle. However, they employ several simplifying assumptions, chiefly the absence of idiosyncratic risk as well as abstracting from borrowing constraints. This implies that their model-free approximation of the labor distortion significantly overstates the progressivity of the net marginal tax rate significantly. In an update to the original study, Cushing (2005) quantifies the spread between effective tax rates on high- and low-income earners to be 19%-points, which reduces to less than 3%-points when including uncertainty and borrowing constraints. Additionally, while certain reform proposals are discussed, none of the above papers carry out any sort of policy experiments based on their findings.

There are several other model-free approximations to the effective tax rate, all of which make assumptions in order to avoid dealing with uncertainty faced by workers. Goda et al. (2007) and Goda et al. (2011) assume that agents behave as if their current earnings are the last earnings in their lives when forming expectations on future benefits. This leads to effective tax rates that are increasing over this lifecycle due to the progressive design of the US pension system. When, instead, explicitly considering workers’ expectations about their future labour supply, the net distortions are decreasing over the lifecycle. In another recent study on the impact of taxes and welfare benefits on labor supply in the UK, Brewer and Shaw (2018) abstract from any inter-temporal considerations (including pension entitle-
ments) altogether.

Finally, there is a sizable quantitative literature on pension reform in OLG lifecycle models. Some of these, including Bagchi (2015), Huggett and Ventura (1999), Huggett and Parra (2010), and Brendler (2016) have included the effect of future pension benefits on labor supply. However, all of these treat the labor supply decision as a "black box", without discussing the effects of modeling decisions on effective labor distortions. As I demonstrate, assumptions on the risk faced by workers, as well as on their ability to borrow can have significant impact on the effective marginal tax rates induced by the pension system, which is in turn likely to influence any verdict on its desirability as an instrument of social insurance.

To summarize, this paper’s contributions are three-fold. First, it is the first study to demonstrate that the net effective tax rates induced by the pension system can be formally derived from workers’ optimality conditions in a lifecycle model with idiosyncratic risk. Second, it quantifies these net distortions in a realistic, calibrated model that allows judgment on the relative importance of influencing factors and modeling decisions. Finally, to my knowledge it is the first to implement a policy experiment aimed at reducing the effective tax rate of the pension system.

3 Deriving Net Effective Tax Rates

3.1 Institutional Framework

In the United States, pensions are administered by the Social Security Administration (SSA). Workers and firms contribute to the budget of the Old Age Survivors and Disability Insurance (OASDI) through a linear payroll tax on all earnings below a certain cap. In 2015, the total statutory rate was set at 6.2% for employers and workers each, coming to a total of 12.4% while the cap on taxable earnings was set at 118,500$ (SSA (2017)). When workers retire, the SSA calculates their Average Indexed Monthly Income (AIME), by taking the indexed average 420 highest earning months (corresponding to 35 years). Indexing is done with the national average wage index in order to ensure that worker’s contributions are not eaten up by inflation and that they profit from productivity growth during their working life. Based on the calculated AIME, the Primary Insurance Amount (PIA) is calculated by applying a highly progressive pension formula. In 2105, the PIA was calculated as 90% of the first 826$ of AIME, plus 32% of the AIME between this first bend-point, and 4,154$, plus 15% of AIME above this second bend-point up to the cap of monthly taxable earnings at 9,875$. Above this cap, the marginal replacement rate drops to zero. Figure 1 plots the

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2Of these, 10.4% are earmarked for the old age pension
average and marginal replacement rates as a function of AIME according to the 2015 SSA rules.

Since the 1977 Social Security Amendments, the marginal replacement rates of 90%, 32%, and 15% have stayed constant, while the two bend-points and the cap on taxable earnings has been indexed to the national average wage.

3.2 The Net Effective Tax Rate in an OLG Model

A key reason why economists care about the net effective rate in the first place is that it is this rate, rather than the statutory rate that measures the distortions of workers’ labor supply decisions. However, instead of trying to approximate the distortions by calculating the net tax rates under simplifying assumptions, it is possible - and arguably simpler - to derive the distortions directly from a realistic model. By taking this approach, we automatically take into account the effects of uncertainty and borrowing constraints.

It is a standard result that the marginal distortion to any decision can be measured by a wedge. This wedge is defined as measuring the difference between an agents marginal rate of substitution (MRS) and their marginal rate of transformation (MRT). Formally, the wedge $\tau$ is defined as

$$\tau = 1 - \frac{MRS}{MRT}.$$ 

If a decision is undistorted, then an agent’s marginal rates of substitution and transformation will be equalized and the wedge $\tau$ will be equal to zero.
In case the decision is distorted, however, the two marginal rates will be different, leading to a non-zero wedge.\footnote{Note that this concept is general enough to also capture the effect of a subsidy which will lead to a negative wedge.}

In the case of the labour wedges that this paper is concerned with, the relevant margin is that of exchanging leisure time for income through labor supply. Thus, the marginal rate of transformation is simply given by the gross wage that workers can earn in the market, given their productivity. On the other hand, their marginal rate of substitution is given by

\[
MRS = \frac{u_l}{u_c},
\]

where \( u_i, i = l, c \) denotes the marginal utility of leisure and consumption, respectively. These two marginal utilities will be determined as part of workers’ optimizing behavior, which is what I will use in order to derive a formulation for the net effective tax rate.

Thus, instead of trying to approximate the effective tax rate in the spirit of most of the literature, this paper will take the approach of solving for an allocation induced by the pension system and then using this allocation to derive the labour wedges, which equal the effective marginal tax rate. This approach is similar in spirit to the approach taken by the New Dynamic Public Finance literature (see for example Farhi and Werning (2013) and Golosov et al. (2016)) that solve for the constrained optimal allocation and use this to characterize optimal labour distortions.

I consider a \( T \) period OLG economy with constant population growth, ex-ante heterogeneity, idiosyncratic productivity risk, and longevity risk. Workers enter the economy at age \( t = 1 \), work until \( t = R - 1 \), and then retire in period \( R \) until they die and exit the economy. They face mortality risk, and have an age specific survival probability \( \psi_t \) of surviving up until age \( t \), conditional on having survived until age \( t - 1 \), and die for sure at age \( T + 1 \). During their working lives, they supply labor to the market, for which they are compensated at a rate \( \theta \), where \( \theta \) denotes a worker’s idiosyncratic productivity. At the start of their lives, each worker observes their ex-ante type \( \theta_1 \), and then draws an idiosyncratic productivity shock \( \theta_t, t \geq 2 \) from a distribution \( F(\theta_t|\theta_{t-1}) \) in each period of their working life.

In each period of their working life, agents observe their current productivity, and make a labor-leisure, as well as a consumption-savings decision. Markets are incomplete, and workers only have access to a risk-less bond which pays an interest rate \( r \). Upon entering retirement, agents cease to supply labor, instead only making a consumption-savings decision until they die and exit the economy. I abstract from any bequest motives, instead the government confiscates all estates and uses them to fund government consumption.
During their working life, agents are subject to a payroll tax $\tau$ on earnings up to the taxable maximum $\hat{y}$, an income tax $T(y)$ on labor income, as well as a flat tax $\tau_a$ on capital income. During retirement, agents receive a pension payment $B(\bar{y}_R)$, which depends on their indexed average earnings up to retirement. Pension contributions are indexed at a rate $i$. It should be noted, that I make one important simplifying assumption with regards to indexing. Specifically, I assume that the returns to pension contributions come from outside of the model instead of from wage growth. While it is possible to derive an implicit return to contributions within the model by including wage growth, this will lead to issues when solving the model, as the standard methods for re-introducing stationarity into a model with wage growth rely on the absence of income effects to the labour supply decision (see for example Fehr and Kindermann (2018)), which cannot be achieved in a setting with a general pension system. The pension schedule $B$ follows the current US schedule presented in the preceding section.

Formally, workers solve the following problem upon observing their productivity draw

$$V_t(a, \bar{y}, \theta) = \max_{c, a', \bar{y}} u \left( c, \frac{\bar{y}}{\theta} \right) + \psi_{t+1} \beta E_t \left[ V_{t+1} \left( a', \bar{y}', \theta' \right) \right],$$

subject to the budget constraint

$$c + a' \leq y - T(y) - \min(y, \hat{y}) \tau + (1 + (1 - \tau_a) r) a, \quad t < R \quad (2)$$

$$c + a' \leq B(\bar{y}) + (1 + (1 - \tau_a) r) a, \quad t \geq R \quad (3)$$

as well as $a_0 = 0$, $a' \geq 0$ and the law of motion for the past average earnings $\bar{y}$

$$\bar{y}_{t+1} = \begin{cases} 
(1 + i) \left( \frac{t-1}{t} \bar{y}_t + \min(y, \hat{y}) \right), & t < R \\
\bar{y}_t, & t \geq R.
\end{cases}$$

Here, $B(\cdot)$ denotes the pension schedule as a function of past average earnings.

We can see directly from this formulation of the agents’ problem that the impact of the pension system on labour supply will be two-fold. On the one hand, there is a marginal tax $\tau$ that is paid on all income below the cap, which will cause a decrease in labor supply. On the other hand, agents realize that working more now will increase their average earnings and hence lead to higher benefits during retirement.

$^4$This formulation represents a good approximation of the current US pension system. The main difference is that in the US system, not all years of earnings count towards the calculation of the AIME, but rather the 35 years with highest earnings.
The first order conditions of an optimizing household lead to the following inter- and intra-temporal optimality conditions

\[
u_c(\cdot) = \beta \psi_{t+1} (1 + (1 - \tau) r) \mathbb{E}_t u_c(c', l') + \mu \tag{4}\]

\[
u_t(\cdot) \frac{1}{\theta} + u_c(\cdot) \left[1 - T'(y) - \mathbbm{1}_{y<y^*} \tau \right] + \beta \psi_{t+1} \mathbb{E}_t \frac{\partial V_{t+1}(\cdot)}{\partial \bar{y}} = 0. \tag{5}\]

While the inter-temporal Euler Equation is not directly affected by the pension system, the intra-temporal condition now contains an extra term, compared to the case without considering the impact of labour supply on future benefits. Specifically, we will have an additional envelope condition that captures the effect of entering the next period with an additional unit of past average income.

We can use the fact that average income does not enter the agents’ budget constraint until the retirement period in order to ‘roll forward’ the envelope condition until period \(R\) to write\[5\]

\[
\frac{\partial V_{t+1}}{\partial \bar{y}} = \frac{1}{R-1} (1+i)^{R-t-1} \beta^{R-t-1} \prod_{q=1}^{R-t} \psi_{t+q} \mathbb{E}_t \left[ B'(\bar{y}_R) \sum_{\tilde{s}=R}^{T} \beta^{\tilde{s}-R} \prod_{\tilde{q}=1}^{\tilde{s}-R} \psi_{\tilde{s}+\tilde{q}} u_c(c_{\tilde{s}}) \right]. \tag{6}\]

Hence, the envelope condition for average income is given by the discounted marginal valuation of benefit payments throughout retirement. Substituting this into the intra-temporal optimality condition, and then in turn substituting this condition into our definition of the effective tax rate

\[
\tau^t_{\text{edge}} = 1 - \frac{M RT}{M RS},
\]

yields the following result:

**Proposition 1.** The effective marginal tax rate of the pension system at age \(t\) is given by

\[
\tau^t_{\text{edge}} = 1 - \mathbb{E}_t \left[ \frac{1}{R-1} (1+i)^{R-t-1} \beta^{R-t-1} \prod_{q=1}^{R-t} \psi_{t+q} \mathbb{E}_t \left[ B'(\bar{y}_R) \sum_{\tilde{s}=R}^{T} \beta^{\tilde{s}-R} \prod_{\tilde{q}=1}^{\tilde{s}-R} \psi_{\tilde{s}+\tilde{q}} \frac{u_c(c_{\tilde{s}})}{u_c(c_t, l_t)} \right] \right].
\]

Thus, the effective marginal tax rate of the pension system has a formulation made up of two parts. The first term describes the instantaneous marginal tax due on any earnings below the cap. The second term, while somewhat unwieldy, denotes the discounted marginal benefits that are due to an extra unit of income in period \(t\). Here, it is important to note that the correct discount factor is the stochastic discount factor, as workers weigh the marginal benefit streams during retirement with their associated marginal utilities of consumption.

\[5\] A detailed proof is given in the appendix
In order to gain a better understanding of the above formula, it is helpful to come back to the leading example of this paper, namely the comparison of two countries, one of which pays pensions as a lump-sum benefit while the other pays pensions as contributions plus interest at the market rate. The case of the first country is simple, as we simply need to set $B'(\cdot) = 0$ in order to see that any household with income below the cap on taxable earnings will see the payroll tax $\tau$ as a full tax. The case for the second country is easier to understand if we make some additional assumptions. Specifically, I assume that the lifecycle lasts only for two periods, there is no population growth or mortality risk, and that workers do not face any borrowing constraints. Given these assumptions, the above formula simplifies to

$$\tau^t_{\text{wedge}} = 1_{y_t < \bar{y}} \left[ \tau - \beta(1 + i)B'(\bar{y}_2) \right],$$

since we can use the Euler equation to substitute in $E \left[ u(c_2) / u(c_1) \right] = 1$. Finally, since benefits are paid in a one-to-one relationship to contributions, we have that $B' = \tau$, and since the pension system is indexed at the market rate, we will have

$$\tau^t_{\text{wedge}} = 1_{y_t < \bar{y}} \left[ \tau - \beta(1 + r)\tau \right] = 0,$$

leading to the anticipated 'zero marginal tax' result.

Clearly, all the above-mentioned simplifying assumptions are unlikely to hold in any realistic model of the lifecycle. Hence, I will devote the following section to decomposing the effective marginal tax rate of the pension system into its components in order to determine the factors influencing it. I will show that the effective marginal tax rate can be separated into five different components: i) inter-generational redistribution, ii) intra-generational redistribution, iii) a demographic component, iv) borrowing constraints, and v) the insurance value of non-linear pensions.

### 4 Decomposing Effective Tax Rates

It is instructive to start this discussion by considering a simplified version of the above model. To this end, consider an economy in which the lifecycle consists of two periods. During the first period agents work, and during the second they are retired and only consume. Agents discount the future at rate $\beta$ and can save using a risk-free bond with interest rate $r = 1 / \beta - 1$. For notational simplicity, I abstract from longevity risk at this stage. Also, I assume that the payroll tax is applied to all earnings. The population

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6 Both of these simplifying assumptions will be dropped when discussing the final result of the section.
in this economy grows at a constant rate, with each generation being \(1 + g\) times the size of the previous one.

In this simplified two-period economy, the effective marginal tax rate is given by

\[
\tau_{\text{wedge}} = \tau - \beta(1 + i) \frac{u_c(c_2)}{u_c(c_1, l_1)} \mathbb{E}(y_1),
\]

as in this case, first period earnings capture the entire history of earnings.

It should be noted that the remainder of this paper will employ a different view on the effective marginal tax rate of the pension system compared to the preceding section. Specifically, until now, we have compared the contemporaneous payroll tax \(\tau\) with marginal benefits paid in the future. In order to better understand the effective tax rates, it is helpful to dispense with this 'dual view', instead expressing the effective marginal tax in terms of replacement rates, only. Specifically, I will use the budget balance of the pension system in order to replace the payroll tax with an expression in terms of the average replacement rate of the pension system.

Using this, we can relate the payroll tax and the pension benefits in the following manner

\[
(1 + g) \int_{\theta_1} \tau y_1 dF(\theta_1) = \int_{\theta_1} \mathbb{E}(y_1) dF(\theta_1),
\]

which immediately implies

\[
\tau = \frac{1}{1 + g} \frac{Y}{\mathbb{E}} = \frac{b}{1 + g},
\]

where \(b\) denotes the average replacement rate.

Thus, workers do not need to pay a payroll tax equal to the average replacement rate since the population is growing over time. This is exactly the effect captured by the third of our effects, the demographic component. The higher the population growth, the lower the payroll tax rate will have to be in order to finance a given level of benefits.

The first component of the effective tax rate, inter-generational redistribution, is driven by the fact that workers use two separate vehicles to save for retirement in this model. One of the vehicles, the pension system, pays a return of \(i\), while the risk free bond that agents could use instead has a return of \(r > i\). The larger this gap in returns, the larger the effective tax rate will be, as agents would have been better off by investing in the private asset instead. This can be interpreted as a form of redistribution between generations insofar as the lack of a market return is caused by the system.

\[\text{Note that it is at this point that the assumption that resources for indexing of pension contributions come from outside of the model takes hold, as the right hand side of the equation contains the un-indexed earnings history rather than the indexed history.}\]
not being pre-funded, implying that all generations are contributing to the windfall gain realized by the initial generation of pensioners, who were able to enjoy a pension without ever contributing to the system.

The second component of the effective tax, namely *intra-generational redistribution* is determined by the relationship between the worker’s individual marginal replacement rate $B'$, and the average replacement rate $b$ in the economy. If a worker faces an individual replacement rate of exactly $b$, for example because the pension system is flat-rate, this component will be equal to zero, as she is not subject to any redistribution within her generation. If she has a replacement rate above $b$, this component will be negative, hence lowering the effective tax rate (and increasing it, respectively, for an individual replacement rate below $b$).

The fourth component of the tax, the effect of *borrowing constraints* represents the fact that pensions are only substitutes for savings in situations where workers would have chosen to save the same amount in the absence of the pension system. A worker who is heavily borrowing constrained will not view pension contributions as savings, but rather as a tax, since she would prefer to consume the income today. Obviously, this component will be zero for all workers who are not borrowing constrained.

The fifth and final component, the *insurance value* of the pension system will be zero for all workers in the simple two-period model, as there is no uncertainty left at the time when agents make their choices. This component will be discussed when we extend the model to three or more periods.

In order to separate out each of the four terms, we can substitute the budget balance condition into the above formula for the effective marginal tax rate, and collect terms. Denoting the Lagrange multiplier on the borrowing constraint by $\mu$, it is simple to transform the above formula for the effective tax rate in order to get the following result:

**Proposition 2** In a two-period OLG economy with ex-ante heterogeneity and constant population growth $g$, the effective marginal tax rate of a PAYGO pension system can be additively separated into four components as

$$
\tau_{\text{wedge}} = r - i \left( \frac{1}{1 + r} \right) \left[ B' \frac{u'(c_2)}{u'(c_1)} \right] \left[ b - \mathbb{E} \right] + \left[ b - \mathbb{E} \right] - \left[ b \frac{g}{1 + g} \right] + \mathbb{E}' \left[ \frac{\mu_1}{u_x(c_1, l_1)} \right].
$$

Considering the *inter-generational redistribution* component of the effective tax, we can see that another way to describe this would be as an opportunity cost of the pension system, as agents could have realized the same marginal benefit by investing in the private security at a higher rate. The larger the difference in rates $r - i$, the larger this component will be.

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8 Full proofs of all results in this section can be found in the appendix.
The intra-generational redistribution component is straightforward, as it is only driven by the difference (in multiple periods, the expected difference) between the private marginal and the average replacement rate in the economy. Here, it is important to note that the average of this component across the population will not necessarily be zero. This is due to the fact that we are comparing a marginal to an average rate. While workers will have to pay for the average rate through their contributions, they will only receive benefits on their last dollar earned at the marginal rate. If the system is highly progressive, as in the US, the average replacement rate will be above the marginal replacement rate for a large portion of the population, leading to a positive intra-generational redistribution component on average.

The demographic component is, in effect, a discount enjoyed by all contributors to the pension system that makes it more affordable to finance an average replacement rate of $b$, the higher the population growth is.

Finally, borrowing constraints increase the effective marginal tax due to the fact that a mandatory pension system may force some workers onto a suboptimal asset accumulation path, which will 'eat up' a fraction $\mu_1 \frac{u_c(c_1, l_1)}{u_c(c_1, l_1)}$ of their private marginal benefit $B^\prime$, since a borrowing constrained worker would have chosen lower savings and more current consumption if given the opportunity.

After the decomposition of the effective tax rate in a two period model, I will now turn to a three-period model with two working periods. As previously stated, the three-period model will introduce the insurance effect to the effective tax rate, arising from the fact that workers may value the insurance across lifetime income histories afforded by a progressive pension system. This effect does not appear in the model with only one working period, as in this case the current income fully characterises the history of lifetime income.

In a three-period OLG model, the effective marginal tax rates will be changing with age and are given by

$$
\tau_{wedge}^1 = \tau - \beta^2 (1 + i)^2 \int_{\theta_2} \frac{u_c(c_3)}{u_c(c_1, l_1)} \frac{1}{2} B^\prime(\bar{y}) dF_{\theta_2}
$$

$$
\tau_{wedge}^2 = \tau - \beta (1 + i) \frac{u_c(c_3)}{u_c(c_2, l_2)} \frac{1}{2} B^\prime(\bar{y})
$$

Once again, we can use budget balance to express the payroll tax in terms of the average replacement rate $b$, as

$$
\tau = b \left( \frac{\bar{Y}}{(1 + g)^2 \int_{\theta_1} y_1 dF(\theta_1) + (1 + g) \int_{\theta_2} y_2 dF(\theta_2)} \right) = b \frac{\bar{Y}}{\sum Y},
$$

with $\bar{Y} = \frac{1}{2} \int_{\theta_1} y_1 dF(\theta_1) + \frac{1}{2} \int_{\theta_2} y_2 dF(\theta_2)$ being average lifetime earnings of a pensioner, and $\sum Y$ being aggregate earnings in the economy at the time.
of pension entry. Note that we can no longer replace this fraction with a simple function of \( g \), as average income may differ between age groups. The principle, however, stays the same with \( g > 0 \) implying that \( \frac{\bar{Y}}{Y} < \frac{1}{2} \).

Once more, we substitute the budget balance condition into the above formulas for the effective tax rate, and employ the same technique as in the previous proposition.

**Proposition 3** In a three period OLG economy with constant population growth, ex-ante heterogeneity, and idiosyncratic risk, the effective marginal tax of a PAYGO pension system can be additively separated into five components as

\[
\tau_{wedge}^1 = \frac{1}{2} \left[ \frac{(1 + r)^2 - (1 + i)^2}{(1 + r)^2} \right] \mathbb{E} \left[ u'(c_3) \right] \mathbb{E} \left[ b - B'^{(\bar{y})} \right] - \mathbb{E} \left[ \sum \frac{1}{2} Y - \bar{Y} \right] + \mathbb{E} \left[ \mu_1 + \mu_2 \right] \mathbb{E} \left[ B' \right] - \text{Cov} \left( \frac{u_c(c_3)}{u_c(c_1, l_1)}, B' \right)
\]

\[
\tau_{wedge}^2 = \frac{1}{2} \left[ \frac{r - i}{1 + r} B' \left( 1 - \frac{\mu_2}{u_c(c_2, l_2)} \right) \right] + \mathbb{E} \left[ b - B'^{(\bar{y})} \right] - \mathbb{E} \left[ \sum \frac{1}{2} Y - \bar{Y} \right] + \mathbb{E} \left[ B' \right] \mathbb{E} \left[ \mu_2 \right] \mathbb{E} \left( u_c(c_2, l_2) \right).
\]

The most notable difference to the two period model is the introduction of the insurance value of the pension system. If the replacement rate is negatively correlated with consumption in the pension period (i.e. pensioners are receiving more marginal income in states of the world where they value it more), this will lead to an additional insurance value of the pension on top of the expected marginal replacement rate \( B' \). This effect only materializes if there is still uncertainty left to be resolved at the time that the labour supply decision is taken. This is why the effect is present in the first period (when the productivity draw of the second period is still uncertain), but not in the second period, when agents know their entire history upon making their labour supply decisions.

A second point of note relates to borrowing constraints. Here, it is important to recognise that not only current borrowing constraints matter, but also workers’ expectations of being borrowing constrained in the future. This will be particularly important in the general model with multiple retirement periods and mortality risk, as it relates to an additional restriction

\footnote{Recall that there are now two generations contributing to the pension payments of the current old instead of just one}
of pensions not discussed previously. Since pensions are paid out as an annuity, agents cannot borrow against them during retirement. As, however, conditional survival probabilities drop sharply towards the end of agents’ lives, it seems likely that they would choose to front-load consumption if given the chance. Since pensions do not provide this flexibility, there will be an effect of borrowing constraints even after all uncertainty with regards to lifetime income has been resolved.

Finally, we also need to take into account that there are now two generations financing benefit payments, implying that each worker will only have to pay half as much to finance the same level of benefits. Thus, in a more general model, the term pre-multiplying the respective effective tax rates will be driven by the ratio of working to retired agents. This has an immediate implication for the policy debate over retirement ages, as increasing the retirement age will decrease the ratio and hence decrease the average effective marginal tax of the pension system (keeping benefits constant).

The three period model captures all salient features of the full model, without burying any insights into the driving forces of effective tax rates below overly complicated notation. However, in order to later conduct a meaningful quantitative analysis of existing pension systems, a similar result for the full model is needed.

The underlying economy changes with regard to the three period model in two ways, both of which will lead to some notational changes which I will point out below. First we need to dispense with the assumption that pensioners are retired for only one period, as this clearly is not the case in reality. Second, any realistic lifecycle model will have to take into account longevity risk. Specifically, I return to the full model introduced in section 3, in which workers face ex-ante heterogeneity, idiosyncratic productivity risk, and longevity risk.

**Proposition 4** In the general model, the effective marginal tax rate of the pension system in period $t$ can be additively decomposed into five components as

$$
\tau_{\text{wedge}}^t = \frac{1}{R-1} \left[ \text{inter-generational redistribution} + \text{intra-generational redistribution} - \text{demographic component} + \text{borrowing constraints} - \text{insurance value} \right],
$$

where

- $\text{inter-generational redistribution}$ represents the transfer of resources from working to retired agents.
- $\text{intra-generational redistribution}$ captures the redistribution within the same generation.
- $\text{demographic component}$ accounts for changes in age distribution.
- $\text{borrowing constraints}$ reflects the ability to borrow against future income.
- $\text{insurance value}$ represents the value of insurance against longevity risk.
with

\[
\text{inter-generational redistribution} = \sum_{t=0}^{T-1} \left( (1 + (1 - \tau_a) r_i)^{i-t} - (1 + i)^{R-t} \right) \beta^{i-t} \Pi_{q=1}^{\bar{t}} \psi_{t+q} \mathbb{E} \left[ \frac{u'(c_t)}{u'(c_\bar{t})} B'(y_R) \right]
\]

\[
\text{intra-generational redistribution} = \sum_{t=0}^{T} \mathbb{E} \left[ b - B'(y_R) \right]
\]

\[
\text{demographic component} = \sum_{i=R}^{T} \left( 1 - (1 + g)^{-(i-R)} \Pi_{q=1}^{R-q} \sum_{Y} \right) b
\]

\[
\text{borrowing constraints} = \sum_{t=0}^{T} \mathbb{E} \left[ B'(y_R) \right] \mathbb{E} \left[ \frac{u'(c_t) - (1 + (1 - \tau_a) r_i)^{i-t} \beta^{i-t} \Pi_{q=1}^{\bar{t}} \psi_{t+q} u'(c_t)}{u'(c_t)} B'(y_R) \right]
\]

\[
\text{insurance value} = \sum_{t=0}^{T} \left( 1 + (1 - \tau_a) r_i \right)^{i-t} \beta^{i-t} \Pi_{q=1}^{\bar{t}} \psi_{t+q} \text{Cov} \left( \frac{u'(c_t)}{u'(c_\bar{t})}, B'(y_R) \right)
\]

All of these terms are straightforward generalizations of their counterparts in the three period model, with notational differences arising from the introduction of multiple pension periods as well as mortality risk.

An important reason given for the presence of public pension systems is the presence of longevity risk, as it protects workers from out-living their savings. If workers value this insurance, we would expect it to be reflected in the marginal valuation of pension benefits and hence in the effective marginal tax rate. This expectation, it turns out, is met, as the effect of this insurance against longevity is, in fact, captured by the demographic component. While, at first, somewhat counterintuitive, this makes sense when we consider that the advantage of annuities over private savings in insuring against longevity comes from the fact that, in the absence of annuities, workers would have to accumulate sufficient assets to cover the maximum possible lifespan. When annuities, or public pensions, are available however, workers only need to accumulate sufficient assets to cover their expected lifespan, as there is no aggregate longevity risk. Thus, the shorter the expected lifespan after entering retirement is (holding the maximum possible lifespan constant), the higher the demographic component will be, in turn leading to a lower effective tax rate.

Following this detailed theoretical analysis of the effective tax rate and its components, the next section will deal with their quantitative analysis. To this end, I will build a lifecycle model which will be calibrated to the US economy. Of specific interest in this section is the question how much the progressive nature of the pension system is reflected in the progressivity of the effective tax rate. Additionally, I will devote some time to discussing the relative sizes of the different components of the effective tax rate, as this will give some insight into how the distortions induced by a pension system may be reduced.
5 Quantitative Analysis

5.1 Setup

In order to analyze the quantitative importance of the pension wedges, I build a partial equilibrium OLG model with uncertain lifetime, ex-ante heterogeneity, idiosyncratic risk, endogenous labour supply, and a consumption savings decision.

Each period, a new cohort is born, whose size is \(1 + n\) times that of the previous cohort, with the population growth rate \(n\) being constant over time. Agents in my model live for a maximum of \(T\) periods. During their lives, they work for the first \(R - 1\) periods, and then retire at age \(R\), from which time on they draw a pension and only consume. They face mortality risk throughout their lives - with the age specific survival probability \(\psi_{t+1}\) of surviving from age \(t\) to \(t + 1\). At age \(T + 1\) agents die for sure and exit the economy. During their life, agents can accumulate assets at an exogenous interest rate \(r\), both to insure against idiosyncratic shocks to labour productivity and to save for retirement. I assume that they do not have any bequest motive.

Given that the demographic characteristics of the economy are stable over time, there will always be a fraction \(\mu_t\) of age \(t\) individuals. Given the population growth rate \(n\) and the survival probabilities \(\psi\), we can relate the relative sizes of cohorts as

\[
\mu_{t+1} = \frac{\psi_{t+1}}{1 + n} \mu_t.
\]

For simplicity, I normalize these relative weights such that \(\sum_{t=1}^{T} \mu_t = 1\).

5.2 The Government

The government employs three different instruments. First, there is a capital gains tax \(\tau_a\). Second, there is a non-linear income tax \(T(y)\) on labour income. These two taxes, together with the confiscation of accidental bequests are employed to fund some exogenous, and wasteful government spending \(G\). Finally, the government runs a balanced social security system.

I model the social security system to be as close as possible to the current US system. Specifically, the government collects a payroll tax \(\tau_{ssc}\) on all earnings below the social security cap \(\hat{y}\). Contrary to the current US system, I model this tax as being exclusively levied on workers, instead of being shared between workers and firms as I am operating in partial equilibrium. During retirement, agents receive a pension as a function \(B\) of their average income \(\bar{y}\). It is important to note that, in keeping with the current US
system, only income up to the cap $\bar{y}$ counts toward calculating $\bar{y}$, i.e. that

$$\bar{y}_t = \frac{1}{\min(t, R-1)} \sum_{s=1}^{\min(t, R-1)} (1 + i)^{\min(t, R-1) - s} \min(y_s, \bar{y}),$$

where $i$ is the indexing applied to workers pension contributions. The function $B$ is identical to the one currently used in the US. Writing the average income in the economy as $\hat{y}$, we have

$$B_t(\bar{y}) = \begin{cases} 0.9 \cdot \bar{y}, & \bar{y} \leq 0.2 \cdot \hat{y} \\ 0.9 \cdot 0.2 \cdot \hat{y} + 0.32 \cdot (\hat{y} - 0.2 \cdot \hat{y}), & 0.2 \cdot \hat{y} < \bar{y} \leq 1.25 \cdot \hat{y} \\ 0.9 \cdot 0.2 \cdot \hat{y} + 0.32 \cdot (1.25 - 0.2) \cdot \hat{y} + 0.15 \cdot (\bar{y} - 1.25 \cdot \hat{y}), & 1.25 \cdot \hat{y} < \bar{y} \leq 2.5 \cdot \hat{y} \\ 0.9 \cdot 0.2 \cdot \hat{y} + 0.32 \cdot (1.25 - 0.2) \cdot \hat{y} + 0.15 \cdot (2.5 - 1.25) \cdot \hat{y}, & 2.5 \cdot \hat{y} < \bar{y} \end{cases}$$

for $t \geq R$, and $B_t(\bar{y}) = 0$ otherwise.

### 5.3 Households

Upon being born, agents draw an initial productivity type $\theta_1$ from the distribution $F(\theta_1)$. For $t > 1$, productivity follows a Markov process with conditional distribution $F_t(\theta_t|\theta_{t-1})$. Thus, there is both ex-ante heterogeneity between households as well as idiosyncratic risk.

During their working lives, agents are faced with wages $\omega_t = \gamma_t \theta_t$, where $\gamma$ describes the common life-cycle component to productivity, while $\theta$ is the agent’s idiosyncratic productivity. In retirement, the life-cycle productivity component drops to zero for all agents immediately implying an exit from the labour force. Upon observing her productivity, the agent makes a labour-leisure and a consumption-saving decision, taking the government’s policy into account. They do so in order to maximize their expected discounted sum of utilities, with the flow utility in each period being given by

$$U_t = \frac{c_t^{1-\sigma}}{1-\sigma} + \xi (1 - l_t)^{1-\nu},$$

where $c_t$ denotes current consumption, $l_t$ denotes the fraction of time devoted to the labour market, and $\xi$ is the relative weight of consumption versus leisure in the utility function that is common to all agents.

As agents take the government’s policies into account when making labour supply decisions, their information set needs to include a statistic used to form expectations on their marginal replacement rate in retirement. In order to be able to write the household problem recursively, I use the simple form of $\bar{y}$ to write a law of motion for it
as
\[
\bar{y}_{t+1} = \begin{cases} 
(1 + i)^{(t-1)\bar{y}_t + \min(y_t, \bar{y})} & 1 < t \leq R - 1 \\
\bar{y}_t & t > R - 1
\end{cases}
\]
and \(\bar{y}_1 = 0\). Thus, the state \(\bar{y}\) describes an agent’s current average past income that was subject to the social security payroll tax.

Denoting the value of an agent at age \(t\) by \(V_t(a, \bar{y}, \theta)\), where \(a\) is their current asset holdings, \(\bar{y}\) their past average income and \(\theta\) their current idiosyncratic shock to productivity, we can write the household problem recursively as
\[
V_t(a, \bar{y}, \theta) = \max_{c, a', l} c^{1-\sigma} \left( \frac{1 - \nu}{1 - \sigma} \right)^{1-\nu} + \beta \psi_{t+1} \int_{\theta'} V_t(a', \bar{y}', \theta') dF(\theta'|\theta)
\]
subject to
\[
c + a' \leq \gamma_l l - \tau_{ssc} \sum_{t=1}^{R-1} \mu_t + \sum_{t=1}^{R-1} \mu_{t-1} \psi_t \int a' d\Lambda(a, \bar{y}, t-1)
\]
as well as \(a_0 = \bar{y}_0 = 0\), and the borrowing constraint \(a' \geq 0\).

The result of the household’s maximization problem will be two age-specific policy functions \(a'(a, \bar{y}, \theta, t)\) and \(l(a, \bar{y}, \theta, t)\).

### 5.4 Equilibrium Definition

A **partial equilibrium** in the OLG economy is defined as two policy functions \(a'(a, \bar{y}, \theta, t)\) and \(l(a, \bar{y}, \theta, t)\), together with a tax schedule \(T\), a payroll tax \(\tau_{ssc}\), and a distribution \(\Lambda(a, \bar{y}, \theta, t)\) such that, given the benefit schedule \(B\) and the interest rate \(r\)

1. The policy functions \(a'(a, \bar{y}, \theta, t)\) and \(l(a, \bar{y}, \theta, t)\) solve the household maximization problem (7).
2. The government runs a balanced budget, i.e.
\[
\sum_{t=1}^{R-1} \mu_t \int \gamma_l l(a, \bar{y}, \theta, t) d\Lambda(a, \bar{y}, \theta, t) + \sum_{t=1}^{T} \mu_{t-1} \psi_t \int (1 - \psi_{t-1}) d\Lambda(a, \bar{y}, \theta, t-1)
\]
\[
+ \sum_{t=1}^{T} \mu_{t-1} \int a'(a, \bar{y}, \theta, t-1) d\Lambda(a, \bar{y}, \theta, t) \geq G
\]
3. The social security system runs a balanced budget, i.e.

\[ \sum_{t=1}^{R-1} \mu_t \int \min(\gamma_t \theta l(a, \bar{y}, \theta, t), \bar{y}) \tau_{ssa} \, d\Lambda(a, \bar{y}, \theta, t) \geq \sum_{t=R}^{T} \mu_t \int B(\bar{y}) \, d\Lambda(a, \bar{y}, \theta, t) \]

4. The distribution \( \Lambda(a, \bar{y}, \theta, t) \) is consistent with individual choice, i.e.

\[ \Lambda(\tilde{a}, \tilde{\bar{y}}, \theta', t + 1) = \int_{X_t} f(\theta' | \theta) \, d\Lambda(a, \bar{y}, \theta, t), \]

where \( X_t = \{(a, \bar{y}, \theta) \text{ s.t. } a'(a, \bar{y}, \theta, t) = \tilde{a} \land \bar{y}'(l(a, \bar{y}, \theta, t)) = \tilde{\bar{y}}\} \), as well as with the initial conditions \( a_0 = \bar{y}_1 = 0 \).

5.5 Parameterization of the Model

I parameterize the model to reflect key aspects of the US economy.

**Demographics**: Agents enter the economy at age 21, and die for sure at age 101. I set the annual population growth rate \( n = 1.04\% \), which coincides with the US population growth rate since the 1960’s. The age specific survival probabilities are taken from the 2009 US life tables for males from Arias (2009). One period in my model corresponds to two years.

**Taxation**: I set the tax rate on capital income to \( \tau_a = 0.2 \) following Nardi et al. (2016). The level of government spending is taken from Huggett and Ventura (1999) and set at 19.5% of average income. In order to include the insurance effect of taxation into my model, I choose a non-linear income tax function \( T(y) = y - \lambda y^{1-\tau} \), which is frequently used in the literature and has been shown to approximate the current US tax system relatively well. I follow Heathcote et al. (2014) in setting \( \tau = 0.151 \) and calibrate \( \lambda \) to balance the government’s budget. Indexing of workers’ wages is done at an annual rate of 0.6%, which is consistent with the average real indexing of social security contributions since 1974\(^{10}\). Finally, the social security payroll tax is determined inside the model to balance the social security budget.

**Productivity**: For the age productivity profile, I take the age profile of earnings for US males, with the numerical values taken from Fehr and Kindermann (2018). For the idiosyncratic productivity process, I assume that \( \log(\theta) \) follows an AR(1) process as

\[ \log(\theta_t) = \rho \log(\theta_{t-1}) + \varepsilon_t, \quad t \geq 2 \]

\(^{10}\)Own calculations based on SSA data
with $\varepsilon \sim N(0, \sigma^2_\varepsilon)$ and $\log(\theta_1) \sim N(0, \sigma^2_0)$. Thus, $\rho$ and $\sigma^2_\varepsilon$ drive the idiosyncratic risk, while $\sigma^2_0$ drives initial heterogeneity. In parameterizing this process, I follow Huggett and Ventura (1999) and set $\rho = 0.985$, $\sigma^2_\varepsilon = 0.015$, and $\sigma^2_0 = 0.27$. For the numerical calculations, I discretise this process with $S = 19$ grid points. 18 of these are spaced evenly around zero between $-4\sigma^2_0$ and $4\sigma^2_0$, while I add an additional ultra-high income state at $5\sigma^2_0$. Finally, I rescale the state space $\Theta$ such that the average income during working life is equal to one.

Preferences: I set the risk preference parameter to $\sigma = 2$, while I set $\nu = 4$. The latter is chosen in order to yield a Frisch-elasticity of labour supply of 0.5 at the average labour supply. The relative weight on leisure in the utility function $\xi$ is calibrated so that agents devote an average of $\bar{l} = 0.33$ of their time endowment to the labour market. The annual interest rate is set to 3.85%, which is the average US real interest rate since the 1960’s (World Bank (2018)). Finally, the discount factor is calibrated so that, in combination with the mortality risk, there is no systematic bias towards borrowing or saving over the lifecycle. This leads to an annual discount factor of 0.98.

5.6 The Pension Wedge in the OLG Economy

Using the above baseline calibration, the equilibrium payroll tax that balances the social security budget is $\tau_{ssc} = 11.3\%$. If, however, one considers the actual distortion induced by the pension system, as measured by the effective marginal, the average distortion in the economy is reduced to 4.3%.

Figure 2 depicts the average effective tax rate over the lifecycle as well as it’s decomposition. Over the course of a worker’s life, the average effective tax in her cohort falls from 8.4% at age 21 to −0.8% at age 63, the last working period before retirement.

This strong decline is predominantly caused by variation in the impact of borrowing constraints over the lifecycle. As one would expect, young agents without sizable asset holdings are far more likely to be borrowing constrained, with the importance of borrowing constraints declining over the lifecycle. Even towards the end of the working life, this component does not disappear, despite the resolution of all idiosyncratic risk. This is due to the fact that agents expect to want to borrow against future pension payments in order to front-load consumption during retirement, which I do not permit in this model. As borrowing constraints decrease over the working life, the private valuation of pension payments increases, leading to a larger inter-generational
redistribution component, as a larger expected benefit implies larger opportunity cost of not financing this benefit through the private asset. Once borrowing constraints stabilize in the mid 40’s, the importance of discounting takes over, causing the inter-generational redistribution component to fall again, as workers have successively less time up until retirement over which to discount. It does not, however, go towards zero at the end of the working life, as workers also discount over retirement years, when the pension system no longer pays any return to contributions, whereas the private asset would still pay the market return $r$.

As previously discussed, the average intra-generational redistribution component is not zero, as workers compare the average replacement rate in the economy with their private expected replacement rate. Due to the progressive nature of the pension system, this difference will be positive on average, and stable over the life cycle. The second component related to redistribution, the insurance value of the progressive pension system reduces the effective tax rate slightly during the first half of the working life, and then tapers off quickly towards zero, as agents’ receive more and more information about their eventual replacement rate, reducing the variance, and hence the insurance value.
of redistribution. It should also be noted that this insurance value is very small compared to the average intra-generational redistribution component, giving a first suggestion that, at least from the point of view of marginal tax rates, a linear pension system may have some advantages over a progressive system. Here it should be noted, however, that this only considers the desirability of marginally more insurance, and not the value of all insurance provided by the system. The way to consider this insurance value is as the marginal valuation agents attach to the insurance provided by the last dollar of their earnings, given all the insurance provided by the pension system through their average earnings.

Finally, the demographic component is sizable and reduces the effective rate by a constant amount for each cohort. As discussed previously, the component includes both the effect of population growth as well as the benefits of annuitisation as workers have to pay lower payroll taxes in order to insure themselves against longevity risk when annuities are available (or in this case provided through public pensions).

Due to the strongly progressive nature of the pension system, a natural question to ask is if the resulting effective tax rate will display a similar degree of progressivity. In order to answer this question, figure 3 displays the average effective tax rate by income group for each cohort. Income groups here are defined the same as in Feldstein and Samwick (1992), in that they correspond to the current "bracket" of the pension system the worker would fall into if she were to earn this amount over her entire life.

Contrary to the previous literature, I find that there is no significant spread between the effective marginal tax rates of low-, medium-, and high-income workers. This, of course, is partially driven by the imperfect correlation of current to lifetime income. As the literature following Feldstein and Samwick (1992) does not consider idiosyncratic risk, current earnings are the same as expected lifetime earnings. Thus, a low income worker at age 21 knows that they will enjoy a marginal replacement rate of 90%, while a worker in a more realistic lifecycle model will need to form expectations about their lifetime income given the presence of idiosyncratic shocks. This, however, cannot explain the second, more surprising insight, namely that the low-income workers do not, in fact, have the lowest effective marginal tax rate for most of their working lives. Given that there is at least some positive correlation between current and lifetime incomes, one would intuitively expect that low income workers should expect to receive the highest replacement
Figure 3: The average effective tax rate over the life-cycle by current income

Note: Pension wedges at each age $t$ are averaged using the conditional distribution $\Lambda(a, \bar{y}, \theta, t|y)$

rates and hence have the lowest effective marginal tax rates. However, as the graph shows, they actually have the highest effective rates over the entire lifecycle. Since this clearly cannot be explained by imperfect correlations between current and lifetime incomes, I will instead turn to the decompositions of these tax rates in order to gain a better understanding.

Considering figures 4 to 6, we can see that while the effective marginal tax rates may not vary strongly with income within each cohort, the decomposition thereof does vary significantly. As we can see, the intragenerational redistribution component does indeed rise significantly with income. This suggests that the progressive nature of the pension system does indeed ‘bleed through’ into the effective marginal tax rates, as workers with high current incomes expect significantly lower marginal replacement rate than those with lower incomes. However, this effect is counteracted, and even overpowered, by the impact of borrowing constraints, which affect low-income workers much more severely than those with current medium or high incomes. High-income workers, in particular, are almost unaffected by borrowing constraints during their working life, instead only caring about their inability to front-load consumption during retirement due to the annuity-like na-
Figure 4: The average effective tax rate over the life-cycle - low income

Note: Pension wedges at each age \( t \) are averaged using the conditional distribution \( \Lambda(a, \bar{y}, \theta, t|y_l) \)

Figure 5: The average effective tax rate over the life-cycle - medium income

Note: Pension wedges at each age \( t \) are averaged using the conditional distribution \( \Lambda(a, \bar{y}, \theta, t|y_m) \)

Figure 6: The average effective tax rate over the life-cycle - high income

Note: Pension wedges at each age \( t \) are averaged using the conditional distribution \( \Lambda(a, \bar{y}, \theta, t|y_h) \)
ture of pension payments. Finally, it should be noted that even for low-income workers, the intra-generational redistribution component is strictly positive over their entire working life, as even their expected replacement rate lies below the average replacement rate in the economy due to imperfect correlation between current and lifetime income.

6 Policy Experiment - Equalizing Replacement Rates

The previous section demonstrates that out of the two effects of non-linear pensions, a distributional component that, on average, increases the effective tax and an insurance component that decreases it, the distributional component is larger by far. This suggests that an intuitive reform targeted at reducing the labour supply distortion arising from the pension system would be to equalize replacement rates by replacing the current pension schedule with a flat schedule that pays the current average replacement rate to each worker regardless of income history. Intuitively, this will remove the intra-generational redistribution component, as each agent now perceives the average replacement rate as their expected marginal replacement rate.

With the above parameterization, the average replacement rate in the economy is 41.9%. Thus, I define the new pension schedule

$$\tilde{B}_t(\bar{y}) = \begin{cases} 0.419 \cdot \min(\bar{y}, \hat{y}) & , t \geq R \\ 0 & , \text{o.w.} \end{cases}$$

Using the same parameterization as previously, I solve the lifecycle model again for this pension schedule. Once again, the average tax rate $1 - \lambda$ and the social security payroll tax $\tau_{ssc}$ will adjust in order to balance the social security, and government budget, respectively.

Figure 7 depicts the average effective tax rate of the pension system under a proportional benefit rule, as well as its decomposition. As we can see, the average effective tax rate lies below that of the current US pension system for every cohort. Averaging over all cohorts, this reform would reduce the average effective rate from 4.3% to 0.7%. This decline in the effective marginal rate causes the aggregate income in

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11 If one were to redo the above graphs in terms of current lifetime, instead of current annual income, the intra-generational redistribution component would, of course, turn negative for low lifetime income workers towards the end of their lives, as their expected marginal replacement rate would move ever closer to 90%, while the average replacement rate stays constant at 42%.
Considering the decomposition of the effective tax rate under a linear benefit rule, we find that, as expected, the intra-generational component as well as the insurance value are zero. Since the former outweighs the latter under the current system, this leads to a net decline in the effective rate. We do, however, also find that the impact of borrowing constraints, as well as of inter-generational redistribution is increased. This is simply due to the fact that both terms are multiplicative in workers’ expectations of their eventual replacement rate. Since this rate increases on average when replacing the strongly progressive with a linear pension system in which everyone faces the average as their marginal replacement rate, so will the two aforementioned components. Finally, the demographic component is unchanged, as this reform will not impact any of the demographics of the model.

It should be noted, that these results do not immediately imply that such a reform would be desirable. The above analysis is only concerned with the effect of such a reform on effective marginal tax rates, and hence, on labour supply. This, however, is not the only effect of
this reform, as the current progressive pension system also provides insurance to workers against bad lifetime income histories. As I have discussed previously, the insurance value under the current system represents the marginal valuation workers attach to the insurance provided by their last dollar of earned income, given the insurance already provided by their infra-marginal earnings through the pension system. Thus, moving from a currently progressive, to a linear pension system will have significant impact on consumption insurance, which is not captured by the marginal valuation of insurance.

In order to arrive at a final conclusion whether such a reform would be desirable, one would need to weigh the loss of consumption insurance over income histories against increased efficiency in the labour market, which, for the moment, I leave to future research.\footnote{A key consideration to keep in mind when conducting such a study is that removing intra-generational redistribution from the pension system will increase overall inequality in the economy. Thus, when deciding on a welfare criterion, it is important to ensure that any welfare gains or losses through this reform are not simply a result of the welfare weights, in the sense that they correct for (or exacerbate) an inefficient level of redistribution in the status quo.}

7 Conclusion

In this paper I theoretically and quantitatively investigate the distortions to labour supply induced by a pay-as-you-go pension system such as the US Social Security System.

In the first part of the paper, I demonstrate that, contrary to the approximation approaches taken by the previous literature, it is possible to calculate the effective marginal tax rates of the pension system directly from the first order conditions of an optimizing household in a heterogeneous agent lifecycle model. This approach has the advantage of allowing a decomposition of the effective tax rate into five principal components. As social security contributions pay a return less than the market rate, there is an inter-generational redistribution component, as all non-initial generations pay towards the windfall gain of the first generation of pensioners that got to enjoy the benefits of a PAYGO pension without making any contributions. Second, due to the nonlinear nature of the US pension system, there is an intra-generational redistribution component. In the presence of mortality risk and population growth, there is a demographic component to the effective marginal tax rate, which is driven by the relative measure of workers to pensioners. Fourth, since pensions and savings are only substitutes for a worker
who is not borrowing constrained, there is a component driven by borrowing constraints. Finally, there is an insurance value to a nonlinear pension, as workers may value pension payments differently, depending on the income history they experienced.

In the second part of the paper, I calibrate the lifecycle to the US economy and quantify the effective marginal tax rates of the US pension system. The average effective rate is significantly below the statutory rate of 11%, at 4%. I find that, despite the highly progressive nature of the US pension system, the effective tax rates do not exhibit the large spread that has been suggested in the previous literature. Indeed, effective marginal rates are non-monotone in income with low income households facing the highest effective tax rate and medium income household the lowest. This relatively narrow spread, however, does not imply that there are no differences by income groups, as the decomposition of the effective marginal tax rate reveals significant heterogeneity. While for low income workers borrowing constraints play a very important role in determining the effective tax rate, the effective rate of high income workers is driven, to a large extend, by intra-generational redistribution. Finally, I note that the average insurance value is quantitatively insignificant, especially when compared to the second component that is directly driven by the progressive structure of the pension system, the intra-generational redistribution component. Since the latter increases the effective tax rate, on average, significantly more than the former reduces it, this suggests that the net effect of the progressive structure of the pension system on labour supply may be negative.

Guided by this last insight, I investigate the effects of a reform that replaces the current, progressive, pension schedule with a linear schedule paying the same average replacement rate to all workers. I find that this reform reduces the average effective tax rate of the pension system by more than 3% points, leading to an increase in effective labour supply of 1% over the status quo.

It is important to once again reiterate that this finding alone is not sufficient to make a convincing case for a reform of the pension system in this spirit. This is due to the fact that removing progressivity from the system will also impact the consumption insurance enjoyed by workers over the course of their lives. In order to adequately weigh these two effects, future research is warranted.
Appendix

Proof of Proposition 1

During retirement, agents solve

$$V_t(a, \bar{y}_R) = \max_{c,a'} u(c) + \beta \psi_{t+1} V_{t+1}(a', \bar{y}_R)$$  \hspace{1cm} (8)

subject to

$$c + a' \leq B(\bar{y}_R) + (1 + (1 - \tau_a) r) a$$  \hspace{1cm} (9)

as well as $a' \geq 0$ and $V_{T+1}(\cdot, \cdot) = 0$.

The resulting first order conditions are

$$u_c(\cdot) - \lambda = 0$$ \hspace{1cm} (10)

$$-\lambda + \beta \lambda' + \mu = 0,$$ \hspace{1cm} (11)

where $\lambda$ denotes the multiplier of the budget constraint, $\mu$ denotes the multiplier on the borrowing constraint and the second equation uses the envelope condition for $a'$.

This leads to the standard inter-temporal optimality condition

$$u_c(c) \geq \psi_{t+1} \beta (1 + (1 - \tau_a)) ru_c(c'),$$ \hspace{1cm} (12)

which holds with equality when the borrowing constraint does not bind.

There is no more risk (except for mortality) in this part of the agents’ life and hence solving this model is equivalent to solving a standard cake eating problem with the trivial extension that agents receive an additional portion of cake (their pension) in each period. Note that upon entering retirement, the statistic of lifetime income stops updating and the relevant state is $\bar{y}_R$ for all retirement periods.

During working life, agents solve

$$V_t(a, \bar{y}, \theta) = \max_{c,a',l} u(c, l) + \psi_{t+1} \beta \mathbb{E}_t [V_{t+1}(a', \bar{y}', \theta')],$$ \hspace{1cm} (13)

subject to the budget constraint

$$c + a' \leq y - T(y) - \min(y, \bar{y})\tau + (1 + (1 - \tau_a) r) a$$  \hspace{1cm} (14)

as well as $a_0 = \bar{y}_0 = 0$, $a' \geq 0$, $l = \frac{\bar{y}}{\theta}$, and the law of motion for $\bar{y}$

$$\bar{y}_{t+1} = \begin{cases} (1 + i) \frac{(t-1)\bar{y}_t + \min(y, \bar{y})}{t}, & t \leq R - 1 \\ \bar{y}_t, & t > R. \end{cases}$$
The first order conditions of this problem are

\[ u_c(\cdot) - \lambda = 0 \quad (15) \]

\[ -\lambda + \beta \psi_{t+1} E_t \frac{\partial V_{t+1}(\cdot)}{\partial a'} + \mu = 0 \quad (16) \]

\[ u_l(\cdot) \frac{1}{\theta} + \lambda [1 - T'(y) - 1_{y<\bar{y} \tau}] + \beta \psi_{t+1} E_t \frac{\partial V_{t+1}(\cdot)}{\partial \bar{y}'} \frac{\partial \bar{y}'}{\partial \bar{y}} = 0. \quad (17) \]

Substituting in for the multiplier on the budget constraint and using the envelope condition \( \frac{\partial V_{t+1}}{\partial a'} = (1 + (1 - \tau)_a) \lambda' \), the system of equations becomes

\[ u_c(\cdot) = \beta \psi_{t+1} (1 + (1 - \tau)_a) \lambda \] 

\[ u_l(\cdot) \frac{1}{\theta} + u_c(\cdot) [1 - T'(y) - 1_{y<\bar{y} \tau}] + \beta \psi_{t+1} E_t \frac{\partial V_{t+1}(\cdot)}{\partial \bar{y}'} \frac{\partial \bar{y}'}{\partial \bar{y}} = 0. \quad (19) \]

While the inter-temporal optimality condition is not affected by the presence of the pension system, the optimal labour-leisure choice now has an added inter-temporal component. This is because current income affects the statistic \( \bar{y} \), which in turn affects the pension to be paid during retirement. In order to make this second formula more intuitive, consider the envelope condition on \( \bar{y} \). For working periods \( t = 1, ..., R - 1 \), we have \( \frac{\partial V_t}{\partial \bar{y}} = (1 + i) \beta \psi_{t+1} E_t \frac{\partial V_{t+1}}{\partial \bar{y}'} \frac{\partial \bar{y}'}{\partial \bar{y}} \), because \( \bar{y} \) does not enter agents’ budget constraints in pre-retirement periods. Continuing to use this, we can forward substitute this equation into itself until we reach the first period of retirement, i.e we have

\[ \frac{\partial V_t}{\partial \bar{y}} = (1 + i)^{R-t} \beta^{R-t} \prod_{q=1}^{R-t} \psi_{t+q} E_t \frac{\partial V_{R}}{\partial \bar{y}_R} \frac{t-1}{R-1}. \quad (20) \]

On the other hand, during a retirement period \( t = R, ..., T \), the envelope condition will be given by

\[ \frac{\partial V_t}{\partial \bar{y}} = \lambda \mathbb{B}(\bar{y}') + \beta \psi_{t+1} \frac{\partial V_{t+1}}{\partial \bar{y}'} \frac{\partial \bar{y}'}{\partial \bar{y}} \]

\[ = \lambda \mathbb{B}'(\bar{y}) + \beta \psi_{t+1} \lambda' \mathbb{B}'(\bar{y}) \frac{\partial \bar{y}'}{\bar{y}} + \beta^2 \psi_{t+1} \psi_{t+2} \frac{\partial V_{t+2}}{\partial \bar{y}''} \frac{\partial \bar{y}''}{\partial \bar{y}'} \frac{\partial \bar{y}'}{\partial \bar{y}} \]

\[ = ... \quad (21) \]

Now, using the fact that during retirement, there is no updating to the state \( \bar{y} \), i.e. \( \frac{\partial \bar{y}'}{\partial \bar{y}} = 1 \), and using (10), we can write this as

\[ \frac{\partial V_t}{\partial \bar{y}} = \mathbb{B}'(\bar{y}) \sum_{s=t}^{T} \beta^{s-t} \prod_{q=1}^{s-t} \psi_{t+q} u_c(c_{s}), \quad (22) \]

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and hence
\[ \frac{\partial V_R}{\partial \bar{y}} = B'(\bar{y}) \sum_{s=R}^{T} \beta^{s-R} \prod_{q=1}^{\bar{y}} \psi_{R+q} u_c(c_s). \] (23)

Substituting this into (20), we get the full envelope condition for \( \bar{y} \), namely
\[ \frac{\partial V_i}{\partial \bar{y}} = \frac{t - 1}{R - 1} (1 + i)^{R - t} \beta^{R - t} \prod_{q=1}^{\bar{y}} \psi_{t+q} \mathbb{E}_t \left[ B'(\bar{y}_R) \sum_{s=R}^{T} \beta^{s-R} \prod_{q=1}^{\tilde{s}} \psi_{R+q} u_c(c_s) \right]. \] (24)

While this may not look particularly intuitive at first sight, it has a lot of parallels to the envelope condition for assets. Primarily, both depend crucially on the marginal utility of consumption in future periods. This should not come as a surprise to readers as pensions can be interpreted as forced savings.

I measure labour distortions as the wedge between the marginal rate of substitution between labour and leisure and the marginal rate of transformation. I.e. I define
\[ \tau^t_{\text{wedge}} = 1 - \frac{MRS}{MRT}, \]
which in my context is given by
\[ \tau^t_{\text{wedge}} = 1 - \frac{u_t(\cdot)}{u_c(\cdot)\theta}. \]

Substituting the first order condition (5), the envelope condition for \( \bar{y} \) into this formula, and using the fact that \( \frac{\partial \bar{y}}{\partial y} = \frac{1 + i}{t} \), will yield the effective marginal tax rate as
\[ \tau^t_{\text{wedge}} = T'(y_t + 1) \left[ \tau - \frac{1}{R - 1} (1 + i)^{R - t} \beta^{R - t} \prod_{q=1}^{\bar{y}} \psi_{t+q} \mathbb{E}_t \left[ B'(\bar{y}_R) \sum_{s=R}^{T} \beta^{s-R} \prod_{q=1}^{\tilde{s}} \psi_{R+q} u_c(c_s) \right] \right]. \] (25)

Thus, the effective marginal tax rate of workers can be separated additively into a tax and a pension component, the second one of which we denote by
\[ \tau^t_{\text{wedge}} = 1_{y_t < \bar{y}} \left[ \tau - \frac{1}{R - 1} (1 + i)^{R - t} \beta^{R - t} \prod_{q=1}^{\bar{y}} \psi_{t+q} \mathbb{E}_t \left[ B'(\bar{y}_R) \sum_{s=R}^{T} \beta^{s-R} \prod_{q=1}^{\tilde{s}} \psi_{R+q} u_c(c_s) \right] \right], \] (26)
which gives the required result.
Proof of Proposition 2

The effective marginal tax rate of the pension system in the two period model is given by

\[ \tau_{\text{wedge}} = \tau - \frac{1 + i}{1 + r} \frac{u_c(c_2)}{u_c(c_1, l_1)} B'(y_1). \]

Replacing the payroll tax with its expression in terms of the average replacement rate, and substituting in the Euler equation \( u_c(c_1, l_1) = u_c(c_2) + \mu \) yields

\[ \tau_{\text{wedge}} = b \frac{1}{1 + g} - \frac{1 + i}{1 + r} B'(y_1) + \frac{1 + i}{1 + r} \frac{u_c(c_2)}{1 + r u_c(c_2) + \mu} B'(y_1) \]
\[ = b \frac{1}{1 + g} - \frac{1 + i}{1 + r} B'(y_1) + \frac{1 + i}{1 + r u_c(c_2) + \mu} B'(y_1) \]
\[ = b \frac{1}{1 + g} - \frac{1 + i}{1 + r} [b + b] + \frac{1 + i}{1 + r u_c(c_2) + \mu} B'(y_1) \]
\[ = b \left[ \frac{1}{1 + g} - 1 + 1 \right] \frac{1 + i}{1 + r} [b - B'] + \frac{1 + i}{1 + r u_c(c_2) + \mu} B'(y_1) \]
\[ = b \frac{r - i}{1 + r} \left[ \frac{1}{1 + g} - b - B' \right] - b \frac{g}{1 + g} + \frac{1 + i}{1 + r} \frac{\mu}{u_c(c_2) + \mu}, \]

Finally, we want to remove the time discounting effects from the term for intra-generational redistribution and the borrowing constraints. We can do this by employing exactly the same technique as above to get

\[ \tau_{\text{wedge}} = b \frac{r - i}{1 + r} + \frac{1 + i}{1 + r} (b - B') - (b - B') + (b - B') - \]
\[ = b \frac{g}{1 + g} + \frac{1 + i}{1 + r} \frac{\mu}{u_c(c_2) + \mu} - B' \frac{\mu}{u_c(c_2) + \mu} + B' \frac{\mu}{u_c(c_2) + \mu} \]
\[ = r \frac{i}{1 + r} \left[ B' \left( 1 - \frac{\mu_1}{u_c(c_1, l_1)} \right) \right] + [b - B'] - b \frac{g}{1 + g} + B' \frac{\mu}{u_c(c_2, l_1)} \]
\[ = r \frac{i}{1 + r} \left[ \frac{u'(c_2)}{u'(c_1)} \right] + [b - B'] - b \frac{g}{1 + g} + B' \frac{\mu}{u_c(c_2, l_1)}, \]

which completes the proof.
Proof of Proposition 3

In a three-period model, the effective marginal tax rates at each age are given by

\[ \tau_{\text{wedge}}^1 = \tau - (1 + i)^2 \beta^2 \int_{\theta_2} \frac{u_c(c_3)}{u_c(c_1, l_1)} \frac{1}{2} B'(\bar{y}) d\theta_2 \]

\[ \tau_{\text{wedge}}^2 = \tau - (1 + i) \beta \frac{u_c(c_3)}{u_c(c_2, l_2)} \frac{1}{2} B'(\bar{y}) \]

Proving the result for the second period tax rate is a simple application of proposition 2, with the growth rate exchanged for the ratio of average lifetime income to total taxable income \( \bar{Y} \). Hence, I will only discuss the result for the first period rate.

Replacing the payroll tax rate with its corresponding expression as a function of the average replacement rate, we get

\[ \tau_{\text{wedge}}^1 = b \frac{\bar{Y}}{\sum Y} - (1 + i)^2 \frac{1}{2} \int_{\theta_2} \frac{u_c(c_3)}{u_c(c_1, l_1)} B'(\bar{y}) d\theta_2. \]

First, we use a standard result from statistics

\[ E[A \cdot B] = E[A]E[B] + \text{Cov}(A, B) \]

in order to separate the idiosyncratic replacement rate and the corresponding utility weights to get

\[ \tau_{\text{wedge}}^1 = b \frac{\bar{Y}}{\sum Y} - (1 + i)^2 \frac{1}{2} \int_{\theta_2} \frac{u_c(c_3)}{u_c(c_1, l_1)} B'(\bar{y}) d\theta_2 - \frac{(1 + i)^2}{(1 + r)^2} \frac{1}{2} \text{Cov} \left( \frac{u_c(c_3)}{u_c(c_1, l_1)}, B'(\bar{y}) \right) \]

\[ = b \frac{\bar{Y}}{\sum Y} - (1 + i)^2 \frac{1}{2} \int_{\theta_2} \frac{u_c(c_3)}{u_c(c_1, l_1)} B'(\bar{y}) d\theta_2 + \frac{(1 + i)^2}{(1 + r)^2} \frac{1}{2} \int_{\theta_2} \text{Cov} \left( \frac{u_c(c_3)}{u_c(c_1, l_1)}, B'(\bar{y}) \right) \]

\[ - (1 + i)^2 \frac{1}{(1 + r)^2} \frac{1}{2} \int_{\theta_2} B'(\bar{y}) d\theta_2 - \frac{(1 + i)^2}{(1 + r)^2} \frac{1}{2} \int_{\theta_2} B'(\bar{y}) d\theta_2 + \frac{(1 + i)^2}{(1 + r)^2} \frac{1}{2} \text{Cov} \left( \frac{u_c(c_3)}{u_c(c_1, l_1)}, B'(\bar{y}) \right). \]

This step allows for a separation of the insurance premium component of the tax rate. Next, we can replace the integral over the utility
weights with its corresponding expression in terms of the borrowing constraints by using the Euler equation

$$\tau_{wedge}^1 = b \sum Y - \frac{(1 + i)^2}{(1 + r)^2} \frac{1}{2} \int_{\theta_2} u_c(c_3) d F_{\theta_2} + \frac{(1 + i)^2}{(1 + r)^2} \frac{1}{2} \int_{\theta_2} \mathbb{E}'(\bar{y}) d F_{\theta_2} + \frac{(1 + i)^2}{(1 + r)^2} \frac{1}{2} \int_{\theta_2} \mathbb{E}'(\bar{y}) d F_{\theta_2} - \frac{(1 + i)^2}{(1 + r)^2} \frac{1}{2} \int_{\theta_2} \mathbb{E}'(\bar{y}) d F_{\theta_2} - \frac{(1 + i)^2}{(1 + r)^2} \frac{1}{2} \int_{\theta_2} \mathbb{E}'(\bar{y}) d F_{\theta_2}$$

This step separates out the effect of borrowing constraints on the effective tax rate. All that remains at this stage is to separate the first line of the above expression into the inter- and intra-generational redistribution component as well as the demographic component.

To separate out the intra-generational redistribution component, we write

$$\tau_{wedge}^1 = b \sum Y - \frac{(1 + i)^2}{(1 + r)^2} \frac{1}{2} \mathbb{E}[\mathbb{B}'] + \frac{(1 + i)^2}{(1 + r)^2} \frac{1}{2} \mathbb{E}[\mu_1 + \mu_2] - \frac{(1 + i)^2}{(1 + r)^2} \frac{1}{2} \mathbb{Cov} \left( \frac{u_c(c_3)}{u_c(c_1, l_1)}, \mathbb{B}'(\bar{y}) \right).$$

Finally, we can separate the inter-generational and the demographic
component as follows

$$\tau_{wedge}^1 = b \left[ \frac{1}{2} \sum_s \frac{Y_s}{2Y} - \frac{1}{2} + \frac{1}{2} - \frac{(1 + i)^2}{(1 + r)^2} \frac{1}{2} \left[ b - \mathbb{E}[\mathbb{B}'] \right] + \frac{(1 + i)^2}{(1 + r)^2} \left( \frac{1}{2} \mathbb{E}[\mu_1 + \mu_2] \mathbb{E}[[c_3]] + \mathbb{E}[[\mu_1 + \mu_2]] \mathbb{E}[[\mathbb{B}']] \right) - \frac{(1 + i)^2}{(1 + r)^2} \left[ b - \mathbb{E}[\mathbb{B}'] \right] - \frac{b}{\sum Y} + \frac{1}{2} \left[ b \frac{(1 + r)^2}{(1 + r)^2} - \frac{1}{2} \frac{1}{2} \mathbb{E}[\mu_1 + \mu_2] \mathbb{E}[[\mathbb{B}']] \right] - \frac{(1 + i)^2}{(1 + r)^2} \frac{1}{2} \mathbb{E}[[c_3]] \mathbb{E}[[\mathbb{B}']] - \frac{(1 + i)^2}{(1 + r)^2} \mathbb{E}[[c_3]] \mathbb{E}[[\mathbb{B}']] \right] .$$

To complete the proof, we now simply separate the time discounting factors, using the same technique as in the previous proof.

**Proof of Proposition 4**

In the general model, the effective marginal tax rate of the pension system in period $t$ is given by

$$\tau^t_{wedge} = 1_{g < \theta} \left[ \tau - \frac{1}{R - 1} R^{t - 1} (1 + i)^R - 1 \sum_{s = R}^{T - 1} \beta^{t - R} \psi_{s + q} \mathbb{E}_t \left[ \mathbb{B}'(\tilde{y}_R) u_c(c_3) \right] \int_{\Omega_{R-s}} \min(y_{R-s}, \tilde{y}) d\theta^{R-s}, \right] ,$$

In order to express the entire marginal tax rate in terms of benefits, rather than the payroll tax, I will first replace the payroll tax by its corresponding expression in terms of the average replacement rate.

Since we have constant population growth, I will normalize each cohort with the size of the cohort of current fresh retirees. Thus, the cohort of workers currently in their last working period is of size $(1 + g)(2 - \psi_R)$, while that of retirees of age $R + 1$ is of size $\frac{1}{1 + g} \psi_{R+1}$.

Thus, the revenue generated by the pension system is

$$\tilde{R} = \tau \sum_{s = 1}^{R - 1} (1 + g)^s \prod_{q = 0}^{s - 1} (2 - \psi_{R-q}) \int_{\Omega_{R-s}} \min(y_{R-s}, \tilde{y}) d\theta^{R-s},$$

where $\Omega$ denotes the set of histories of $\theta$, which together with the starting conditions on the states for assets and average income fully characterise the state space.

On the other hand, the spending side of the pension system is given by

$$\int_{\Omega_R} \mathbb{B}(\tilde{y}_R) d\theta^{R} \left[ \sum_{q = 1}^{T - 1} (1 + g)^{q - \delta} \prod_{q = 1}^{T - 1} \psi_{R+q} \right] .$$
which is simply the total pension benefits per cohort multiplied with the total measure of pensioners relative to ‘fresh’ retirees, taking into account population growth and mortality risk.

Using the budget balance equation \( \tilde{R} = \tilde{S} \), we get

\[
\tau = \frac{\int_{\Omega_R} \mathbb{E}(\tilde{y}_R)d\theta R \left[ \sum_{\tilde{s}=0}^{T-R} (1 + g)^{\tilde{s}} \prod_{\tilde{q}=1}^{\tilde{s}} \psi_{R+\tilde{q}} \right]}{\sum_{s=1}^{R-1} (1 + g)^s \prod_{q=0}^{s-1} (2 - \psi_{R-q}) \int_{\Omega_{R-s}} \min(y_{R-s}, \tilde{y})d\theta R^{R-s}}.
\]

In order to get a meaningful replacement rate, I will normalize the above with the average taxable lifetime income of the current generation of newly retired agents

\[
\tau = \frac{1}{R-1} \frac{\int_{\Omega_R} \mathbb{E}(\tilde{y}_R)d\theta R}{\sum_{s=1}^{R-1} \sum_{t=1}^{T-1} \int_{\Omega_t} \min(y_t, \tilde{y})d\theta t} \left[ \sum_{\tilde{s}=0}^{T-R} (1 + g)^{\tilde{s}} \prod_{\tilde{q}=1}^{\tilde{s}} \psi_{R+\tilde{q}} \right] \frac{\sum_{s=1}^{R-1} \sum_{t=1}^{T-1} \int_{\Omega_t} \min(y_t, \tilde{y})d\theta t}{\sum_{s=1}^{R-1} (1 + g)^s \prod_{q=0}^{s-1} (2 - \psi_{R-q}) \int_{\Omega_{R-s}} \min(y_{R-s}, \tilde{y})d\theta R^{R-s}}.
\]

Here, the term on the first line is the average replacement rate of the pension system, the term on the second line is the ratio of the total taxable income of current fresh retirees over their lifecycle to the taxable income to the economy right now. This ratio will generally be smaller than one due to population growth as well as mortality risk. Finally, the term on the third line denotes to total measure of pensioners in the economy. Denoting the average replacement rate by \( b \), the total taxable income of current fresh pensioners by \( \sum \tilde{Y} \), and the total taxable income by \( \sum Y \), we can write

\[
\tau = b \frac{1}{R-1} \left[ \sum_{\tilde{s}=0}^{T-R} (1 + g)^{\tilde{s}} \prod_{\tilde{q}=1}^{\tilde{s}} \psi_{R+\tilde{q}} \right] \frac{\sum \tilde{Y}}{\sum Y}.
\]

Just as in the proof for the previous result, we will use the identity

\[
\mathbb{E}[A \cdot B] = \mathbb{E}[A] \cdot \mathbb{E}[B] + Cov(A, B)
\]

to separate out the insurance component from the formulation for the effective tax rate. To keep the notation as concise as possible, I will
substitute out for $\tau$ at a later date.

\[
\tau_{\text{wedge}}^t = \mathbb{1}_{y \leq y} \left[ \tau - \frac{1}{R-1} (1 + \hat{\imath}) R^{-t} \beta R^{-t} \prod_{q=1}^{R-1} \psi_{t+q} \sum_{\hat{s}=R}^{T} \beta^{\hat{s}-R} \prod_{\hat{q}=1}^{\hat{s}-R} \psi_{R+\hat{q}} \right. \\
\cdot \mathbb{E} \left[ \mathbb{B}'(\tilde{y} R) \right] \mathbb{E} \left[ \frac{u_c(c_3)}{u_c(c_1, l_t)} \right] \\
- \frac{1}{R-1} (1 + \hat{\imath}) R^{-t} \beta R^{-t} \prod_{q=1}^{R-1} \psi_{t+q} \sum_{\hat{s}=R}^{T} \beta^{\hat{s}-R} \prod_{\hat{q}=1}^{\hat{s}-R} \psi_{R+\hat{q}} \cdot \mathbb{E} \mathbb{B}'(\tilde{y} R) \left. \mathbb{E} \left[ \frac{u_c(c_3)}{u_c(c_1, l_t)} \right] \right].
\]

In a next step, I will separate out the expected replacement rate from the borrowing constraints

\[
\tau_{\text{wedge}}^t = \mathbb{1}_{y \leq y} \left[ \tau - \frac{1}{R-1} (1 + \hat{\imath}) R^{-t} \beta R^{-t} \prod_{q=1}^{R-1} \psi_{t+q} \sum_{\hat{s}=R}^{T} \beta^{\hat{s}-R} \prod_{\hat{q}=1}^{\hat{s}-R} \psi_{R+\hat{q}} \right. \\
\cdot \mathbb{E} \left[ \mathbb{B}'(\tilde{y} R) \right] \mathbb{E} \left[ \frac{u_c(c_3)}{u_c(c_1, l_t)} \right] \\
+ \frac{1}{R-1} (1 + \hat{\imath}) R^{-t} \beta R^{-t} \prod_{q=1}^{R-1} \psi_{t+q} \sum_{\hat{s}=R}^{T} \beta^{\hat{s}-R} \prod_{\hat{q}=1}^{\hat{s}-R} \psi_{R+\hat{q}} \cdot \mathbb{E} \left[ \mathbb{B}'(\tilde{y} R) \right] \mathbb{E} \left[ \frac{u_c(c_3)}{u_c(c_1, l_t)} \right] \\
- \frac{1}{R-1} (1 + \hat{\imath}) R^{-t} \beta R^{-t} \prod_{q=1}^{R-1} \psi_{t+q} \sum_{\hat{s}=R}^{T} \beta^{\hat{s}-R} \prod_{\hat{q}=1}^{\hat{s}-R} \psi_{R+\hat{q}} \cdot \mathbb{E} \left[ \mathbb{B}'(\tilde{y} R) \right] \mathbb{E} \left[ \frac{u_c(c_3)}{u_c(c_1, l_t)} \right] \\
- \frac{1}{R-1} (1 + \hat{\imath}) R^{-t} \beta R^{-t} \prod_{q=1}^{R-1} \psi_{t+q} \sum_{\hat{s}=R}^{T} \beta^{\hat{s}-R} \prod_{\hat{q}=1}^{\hat{s}-R} \psi_{R+\hat{q}} \cdot \mathbb{E} \left[ \mathbb{B}'(\tilde{y} R) \right] \mathbb{E} \left[ \frac{u_c(c_3)}{u_c(c_1, l_t)} \right] \\
\left. \right].
\]

Following this, I will separate out the effects of intra-temporal redis-

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\[ \tau^t_{\text{wedge}} = \mathbb{1}_{y \leq \bar{y}} \left[ \tau - \frac{1}{R - 1} (1 + i)^{R-t} \beta^{R-t} \prod_{q=1}^{R-t} \psi_{t+q} \sum_{\bar{s} = R}^{T} \beta^{\bar{s}-R} \prod_{q=1}^{\bar{s}-R} \psi_{\bar{s}+q} \mathbb{E} \left[ b \right] \right. \\
+ \frac{1}{R - 1} (1 + i)^{R-t} \beta^{R-t} \prod_{q=1}^{R-t} \psi_{t+q} \sum_{\bar{s} = R}^{T} \beta^{\bar{s}-R} \prod_{q=1}^{\bar{s}-R} \psi_{\bar{s}+q} \mathbb{E} \left[ b - \mathbb{E}'(\bar{y}_R) \right] \mathbb{E} \left[ u_e(c_\ell, l_t) - u_e(c_k) \right] / u_e(c_\ell, l_t) \right] \\
- \frac{1}{R - 1} (1 + i)^{R-t} \beta^{R-t} \prod_{q=1}^{R-t} \psi_{t+q} \sum_{\bar{s} = R}^{T} \beta^{\bar{s}-R} \prod_{q=1}^{\bar{s}-R} \psi_{\bar{s}+q} \mathbb{E} \left[ \mathbb{E}'(\bar{y}_R) - \mathbb{E}'(\bar{y}_R) \right] \mathbb{E} \left[ u_e(c_\ell, l_t) - u_e(c_k) \right] / u_e(c_\ell, l_t) \right] .
\]

Now we are ready to separate out the demographic component by substituting \( \tau \) with its corresponding expression in terms of the average
replacement rate.

\[
\tau^t_{\text{wedge}} = \mathbb{1}_{y \leq \bar{y}} \left[ \frac{b}{R - 1} \left[ \sum_{\bar{s} = 0}^{T - R} (1 + g)^{-\bar{s}} \prod_{q=1}^{\bar{s} - R} \psi_{R + \bar{q}} \right] \sum_{\bar{s} = T}^{T} \frac{\bar{Y}}{\sum_{\bar{s} = R}^{T}} + \sum_{\bar{s} = R}^{T} b - \sum_{\bar{s} = R}^{T} b \right] \\
- \frac{1}{R - 1} (1 + i)^{R - t} \beta^{R - t} \prod_{q=1}^{R - t} \psi_{t+q} \sum_{\bar{s} = R}^{T} \beta^{\bar{s} - R} \prod_{q=1}^{\bar{s} - R} \psi_{R + \bar{q}} \beta^{t - \bar{s}} \\
+ \frac{1}{R - 1} (1 + i)^{R - t} \beta^{R - t} \prod_{q=1}^{R - t} \psi_{t+q} \sum_{\bar{s} = R}^{T} \beta^{\bar{s} - R} \prod_{q=1}^{\bar{s} - R} \psi_{R + \bar{q}} b \\
+ \frac{1}{R - 1} (1 + i)^{R - t} \beta^{R - t} \prod_{q=1}^{R - t} \psi_{t+q} \sum_{\bar{s} = R}^{T} \beta^{\bar{s} - R} \prod_{q=1}^{\bar{s} - R} \psi_{R + \bar{q}} \bar{b} \\
- \frac{1}{R - 1} (1 + i)^{R - t} \beta^{R - t} \prod_{q=1}^{R - t} \psi_{t+q} \sum_{\bar{s} = R}^{T} \beta^{\bar{s} - R} \prod_{q=1}^{\bar{s} - R} \psi_{R + \bar{q}} \bar{b} \right].
\]

\[
\tau^t_{\text{wedge}} = \frac{1}{R - 1} \left[ \sum_{\bar{s} = R}^{T} 1 - (1 + i)^{R - t} \beta^{R - t} \prod_{q=1}^{R - t} \psi_{t+q} \sum_{\bar{s} = R}^{T} \beta^{\bar{s} - R} \prod_{q=1}^{\bar{s} - R} \psi_{R + \bar{q}} \beta^{t - \bar{s}} \\
- \sum_{\bar{s} = R}^{T} 1 - \sum_{\bar{s} = 0}^{T - R} (1 + g)^{-\bar{s}} \psi_{R + q} \sum_{\bar{s} = R}^{T} \frac{\bar{Y}}{\sum_{\bar{s} = R}^{T}} \right] b \\
- \frac{1}{R - 1} (1 + i)^{R - t} \beta^{R - t} \prod_{q=1}^{R - t} \psi_{t+q} \sum_{\bar{s} = R}^{T} \beta^{\bar{s} - R} \prod_{q=1}^{\bar{s} - R} \psi_{R + \bar{q}} b \\
+ \frac{1}{R - 1} (1 + i)^{R - t} \beta^{R - t} \prod_{q=1}^{R - t} \psi_{t+q} \sum_{\bar{s} = R}^{T} \beta^{\bar{s} - R} \prod_{q=1}^{\bar{s} - R} \psi_{R + \bar{q}} \beta^{t - \bar{s}} \\
+ \frac{1}{R - 1} (1 + i)^{R - t} \beta^{R - t} \prod_{q=1}^{R - t} \psi_{t+q} \sum_{\bar{s} = R}^{T} \beta^{\bar{s} - R} \prod_{q=1}^{\bar{s} - R} \psi_{R + \bar{q}} b \\
- \frac{1}{R - 1} (1 + i)^{R - t} \beta^{R - t} \prod_{q=1}^{R - t} \psi_{t+q} \sum_{\bar{s} = R}^{T} \beta^{\bar{s} - R} \prod_{q=1}^{\bar{s} - R} \psi_{R + \bar{q}} \beta^{t - \bar{s}} \right].
\]

In a final step, I separate out the inter-generational redistribution component, by collecting the discounting factors, indexing terms, and survival probabilities from the intra-generational, borrowing constraint,
and insurance components.

\[
\tau^t_{\text{wedge}} = \frac{1}{R - 1} \left[ \sum_{s=R}^{T} 1 - (1 + i)^{R-t} \beta^{R-t} \prod_{q=1}^{R-t} \psi_{t+q} \sum_{s=R}^{T} \beta^{s-R} \prod_{q=1}^{s-R} \psi_{R+q} \right] b + (1 + i)^{R-t} \beta^{R-t} \prod_{q=1}^{R-t} \psi_{t+q} \sum_{s=R}^{T} \beta^{s-R} \prod_{q=1}^{s-R} \psi_{R+q} \mathbb{E} \left[ b - \mathbb{B}'(\bar{y}_R) \right] \\
- \sum_{s=R}^{T} \mathbb{E} \left[ b - \mathbb{B}'(\bar{y}_R) \right] \sum_{s=R}^{T} \mathbb{E} \left[ b - \mathbb{B}'(\bar{y}_R) \right] \\
- \left[ \sum_{s=R}^{T} 1 - \sum_{s=0}^{T} (1 + g)\bar{s} \prod_{q=1}^{\bar{s}} \psi_{R+q} \sum_{s=0}^{\bar{s}} \mathbb{Y} \right] b \\
+ (1 + i)^{R-t} \beta^{R-t} \prod_{q=1}^{R-t} \psi_{t+q} \sum_{s=R}^{T} \beta^{s-R} \prod_{q=1}^{s-R} \psi_{R+q} \cdot \mathbb{E} \left[ \mathbb{B}'(\bar{y}_R) \right] \mathbb{E} \left[ \frac{u_c(c_t, l_t) - u_c(c_s)}{u_c(c_t, l_t)} \right] \\
- \sum_{s=R}^{T} \mathbb{E} \left[ \mathbb{B}'(\bar{y}_R) \right] \mathbb{E} \left[ \frac{u_c(c_t, l_t) - u_c(c_s)}{u_c(c_t, l_t)} \right] + \sum_{s=R}^{T} \mathbb{E} \left[ \mathbb{B}'(\bar{y}_R) \right] \mathbb{E} \left[ \frac{u_c(c_t, l_t) - u_c(c_s)}{u_c(c_t, l_t)} \right] \\
- (1 + i)^{R-t} \beta^{R-t} \prod_{q=1}^{R-t} \psi_{t+q} \sum_{s=R}^{T} \beta^{s-R} \prod_{q=1}^{s-R} \psi_{R+q} \text{Cov} \left( \mathbb{B}'(\bar{y}_R), \frac{u_c(c_s)}{u_c(c_t, l_t)} \right) \\
- \sum_{s=R}^{T} \text{Cov} \left( \mathbb{B}'(\bar{y}_R), \frac{u_c(c_s)}{u_c(c_t, l_t)} \right) + \sum_{s=R}^{T} \text{Cov} \left( \mathbb{B}'(\bar{y}_R), \frac{u_c(c_s)}{u_c(c_t, l_t)} \right) .
\]

In order to simplify this expression, I rewrite the following two sums

\[
\sum_{s=R}^{T} 1 - (1 + i)^{R-t} \beta^{R-t} \prod_{q=1}^{R-t} \psi_{t+q} \sum_{s=R}^{T} \beta^{s-R} \prod_{q=1}^{s-R} \psi_{R+q} \\
= \sum_{s=R}^{T} \left[ 1 - (1 + i)^{R-t} \beta^{s-t} \prod_{q=1}^{\bar{s}} \psi_{t+q} \right] \\
\sum_{s=R}^{T} 1 - \sum_{s=0}^{T} (1 + g)\bar{s} \prod_{q=1}^{\bar{s}} \psi_{R+q} \sum_{s=0}^{\bar{s}} \mathbb{Y} \sum_{s=0}^{\bar{s}} \mathbb{Y} \\
= \sum_{s=R}^{T} \left[ 1 - (1 + g)^{-(s-R)} \beta^{s-R} \prod_{q=1}^{s-R} \psi_{R+q} \sum_{s=0}^{\bar{s}} \mathbb{Y} \sum_{s=0}^{\bar{s}} \mathbb{Y} \right] .
\]
Then the above becomes

\[
\tau_{\text{wedge}}^t = \frac{1_{y \leq \bar{y}}}{R - 1} \left[ \sum_{\bar{s} = R}^{\bar{s} - \bar{t}} \psi_{t+q} \right] \cdot \left[ \mathbb{E} [B' (\bar{y}_R)] \left( 1 - \mathbb{E} \left[ \frac{u_c(c_t, l_t) - u_c(c_{\bar{s}})}{u_c(c_t, l_t)} \right] \right) \right]
\]

\[
+ \text{Cov} \left( B' (\bar{y}_R), \frac{u_c(c_{\bar{t}})}{u_c(c_t, l_t)} \right)
\]

\[
+ \sum_{\bar{s} = R}^{T} \mathbb{E} [b - B' (\bar{y}_R)]
\]

\[
- \sum_{\bar{s} = R}^{\bar{s} - R} \left[ 1 - (1 + g)^{-(\bar{s} - R)} \prod_{q=1}^{\bar{s} - \bar{t}} \psi_{R+q} \sum_{\bar{Y}} \bar{Y} \right] b
\]

\[
+ \sum_{\bar{s} = R}^{T} \mathbb{E} [B' (\bar{y}_R)] \mathbb{E} \left[ \frac{u_c(c_t, l_t) - u_c(c_{\bar{s}})}{u_c(c_t, l_t)} \right]
\]

\[
- \sum_{\bar{s} = R}^{T} \text{Cov} \left( B' (\bar{y}_R), \frac{u_c(c_{\bar{s}})}{u_c(c_t, l_t)} \right)
\].

While the above formulation already separates the effective marginal tax rate into its five components, their interpretation does not yet carry over one-to-one from the case with a single pension period without mortality. This is due to the fact that in this case, the simple identity \( \beta(1 + r) = 1 \) no longer holds. Thus, the fourth term in the above equation can no longer be interpreted directly as a borrowing constraint, since the Euler equation is no longer given by \( u_c(c_t) \geq \mathbb{E} [u_c(c_{t+1})] \), but rather by

\[
u_c(c_t) \geq (1 + (1 - \tau_a)r) \beta \psi_t \mathbb{E} [u_c(c_{t+1})].
\]

Thus, in order to get the correct formulation of the borrowing con-
strains, I transform the above as follows

\[ \tau_{\text{wedge}}^t = \frac{1}{R} \left[ \sum_{\bar{s} = R}^{T} \left( 1 - (1 + \tau)\bar{s} \beta^t \prod_{q=1}^{\bar{s} - t} \psi_{t+q} \right) \right] \cdot \left[ \mathbb{E} \left[ \mathbb{B}'(\bar{y}_R) \right] \left( 1 - \mathbb{E} \left[ \frac{u_c(c_t, l_t) - u_c(c_{\bar{s}})}{u_c(c_t, l_t)} \right] \right) \right] 

+ \sum_{\bar{s} = R}^{T} \mathbb{E} \left[ b - \mathbb{B}'(\bar{y}_R) \right] 

- \sum_{\bar{s} = R}^{T} \left[ 1 - (1 + g)^{-(\bar{s} - R)} \prod_{q=1}^{\bar{s} - R} \sum_{Y=1}^{\bar{s} - Y} \right] b 

+ \sum_{\bar{s} = R}^{T} \mathbb{E} \left[ \mathbb{B}'(\bar{y}_R) \right] \mathbb{E} \left[ \frac{u_c(c_t, l_t) - u_c(c_{\bar{s}})}{u_c(c_t, l_t)} \right] 

+ \sum_{\bar{s} = R}^{T} \mathbb{E} \left[ \mathbb{B}'(\bar{y}_R) \right] \mathbb{E} \left[ \frac{u_c(c_t, l_t) - (1 + (1 - \tau_0)\tau_0)\bar{s}}{u_c(c_t, l_t)} \prod_{q=1}^{\bar{s} - t} \psi_{t+q} u_c(c_{\bar{s}}) \right] 

- \sum_{\bar{s} = R}^{T} \mathbb{E} \left[ \mathbb{B}'(\bar{y}_R) \right] \mathbb{E} \left[ \frac{u_c(c_t, l_t) - (1 + (1 - \tau_0)\tau_0)\bar{s}}{u_c(c_t, l_t)} \prod_{q=1}^{\bar{s} - t} \psi_{t+q} u_c(c_{\bar{s}}) \right] 

- \sum_{\bar{s} = R}^{T} \text{Cov} \left( \mathbb{B}'(\bar{y}_R), \frac{u_c(c_{\bar{s}})}{u_c(c_t, l_t)} \right) \right], 

which can be simplified to
\[ \tau^t_{\text{wedge}} = \frac{1}{R - 1} \sum_{\tilde{s}=R}^{T} \left[ 1 - (1 + \bar{\bar{y}} R - t)^{\tilde{s} - t} \prod_{q=1}^{\tilde{s} - t} \psi_{t+q} \right] \cdot \left[ \mathbb{E} \left[ B'(\bar{y}_R) \right] \right] \cdot \left[ 1 - \mathbb{E} \left[ \frac{u_c(c_t, l_t) - u_c(c_{\tilde{s}})}{u_c(c_t, l_t)} \right] \right] \\
+ Cov \left( B'(\bar{y}_R), \frac{u_c(c_{\tilde{s}})}{u_c(c_t, l_t)} \right) \\
+ \sum_{\tilde{s}=R}^{T} \mathbb{E} \left[ b - B'(\bar{y}_R) \right] \\
- \sum_{\tilde{s}=R}^{T} \left[ 1 - (1 + g)^{-(\tilde{s} - R)} \prod_{q=1}^{\tilde{s} - R} \psi_{R+q} \sum_{\sum_{Y}} \right] b \\
+ \sum_{\tilde{s}=R}^{T} \mathbb{E} \left[ B'(\bar{y}_R) \right] \mathbb{E} \left[ \frac{u_c(c_t, l_t) - (1 + (1 - \tau_a) r)^{\tilde{s} - t} \beta^{\tilde{s} - t} \prod_{q=1}^{\tilde{s} - t} \psi_{t+q} u_c(c_{\tilde{s}})}{u_c(c_t, l_t)} \right] \\
+ \sum_{\tilde{s}=R}^{T} \left[ (1 + (1 - \tau_a) r)^{\tilde{s} - t} \beta^{\tilde{s} - t} \prod_{q=1}^{\tilde{s} - t} \psi_{t+q} - 1 \right] \mathbb{E} \left[ B'(\bar{y}_R) \right] \mathbb{E} \left[ \frac{u_c(c_{\tilde{s}})}{u_c(c_t, l_t)} \right] \\
- \sum_{\tilde{s}=R}^{T} Cov \left( B'(\bar{y}_R), \frac{u_c(c_{\tilde{s}})}{u_c(c_t, l_t)} \right). \]

In addition to adjusting the borrowing constraint, we will also have to adjust the insurance value. In this term, we are normalizing the marginal utility of consumption in period \( \tilde{s} \) with the current marginal utility of consumption in period \( t \). However, since we now have mortality risk as well as capital gains taxation, the correct Euler equation to use when comparing marginal utilities is actually

\[ u_c(c_t, l_t) \geq (1 + (1 - \tau_a) r)^{\tilde{s} - t} \beta^{\tilde{s} - t} \prod_{q=1}^{\tilde{s} - t} \psi_{t+q} u_c(c_{\tilde{s}}). \]
Hence, we rewrite the above as

\[
\tau_{\text{wedge}} = \frac{1_{y \leq \tilde{y}}}{R} \left[ \sum_{s = R}^{T} \left( 1 - (1 + i)^{R-t} \beta^{s-t} \prod_{q=1}^{s-t} \psi_{l+q} \right) \right] \cdot \left[ \mathbb{E} \left[ B'(\tilde{y}_R) \right] \left( 1 - \mathbb{E} \left[ \frac{u_c(c_t, l_t) - u_c(c_t)}{u_c(c_t, l_t)} \right] \right) \right]
\]

\[
+ \sum_{s = R}^{T} \mathbb{E} \left[ B'(\tilde{y}_R) \right] \mathbb{E} \left[ \frac{u_c(c_t, l_t)}{u_c(c_t)} \right]
\]

\[
+ \sum_{s = R}^{T} \mathbb{E} \left[ B'(\tilde{y}_R) \right] \mathbb{E} \left[ \frac{u_c(c_t, l_t)}{u_c(c_t, l_t)} \right]
\]

\[
- \sum_{s = R}^{T} \left[ 1 - (1 + g)^{s-R} \prod_{q=1}^{s-R} \psi_{l+q} - 1 \right] \mathbb{E} \left[ B'(\tilde{y}_R) \right] \mathbb{E} \left[ \frac{u_c(c_t, l_t)}{u_c(c_t, l_t)} \right]
\]

\[
+ \sum_{s = R}^{T} \left[ (1 + (1 - \tau_a) r)^{s-t} \beta^{s-t} \prod_{q=1}^{s-t} \psi_{l+q} - 1 \right] \mathbb{E} \left[ B'(\tilde{y}_R) \right] \mathbb{E} \left[ \frac{u_c(c_t)}{u_c(c_t, l_t)} \right]
\]

\[
- \sum_{s = R}^{T} \mathbb{E} \left[ B'(\tilde{y}_R) \right] \mathbb{E} \left[ \frac{u_c(c_t)}{u_c(c_t, l_t)} \right]
\]

\[
+ \sum_{s = R}^{T} \left[ (1 + (1 - \tau_a) r)^{s-t} \beta^{s-t} \prod_{q=1}^{s-t} \psi_{l+q} \right] \mathbb{E} \left[ B'(\tilde{y}_R) \right] \mathbb{E} \left[ \frac{u_c(c_t)}{u_c(c_t, l_t)} \right] \mathbb{E} \left[ \frac{u_c(c_t)}{u_c(c_t, l_t)} \right]
\]

\[
- \sum_{s = R}^{T} \left[ (1 + (1 - \tau_a) r)^{s-t} \beta^{s-t} \prod_{q=1}^{s-t} \psi_{l+q} \right] \mathbb{E} \left[ B'(\tilde{y}_R) \right] \mathbb{E} \left[ \frac{u_c(c_t)}{u_c(c_t, l_t)} \right] \mathbb{E} \left[ \frac{u_c(c_t)}{u_c(c_t, l_t)} \right],
\]

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which can be simplified to

$$\tau^t_{\text{wedge}} = \frac{1}{R-1} \sum_{s=R}^{T} [(1 + (1 - \tau_a) r)^{s-t} - (1 + i)^{R-t}] \beta^{s-t} \prod_{q=1}^{s-t} \psi_{t+q} \cdot \left[ \mathbb{E} \left[ \mathbb{B}'(\bar{y}_R) \right] \left( 1 - \mathbb{E} \left[ \frac{u_c(c_s)}{u_c(c_t, l_t)} \right] \right) \right]$$

$$+ \text{Cov} \left( \mathbb{B}'(\bar{y}_R), \frac{u_c(c_s)}{u_c(c_t, l_t)} \right)$$

$$+ \sum_{s=R}^{T} \mathbb{E} \left[ b - \mathbb{B}'(\bar{y}_R) \right]$$

$$- \sum_{s=R}^{T} \left[ 1 - (1 + g)^{-(s-R)} \prod_{q=1}^{s-R+\bar{Y}} \psi_{R+q} \sum_{\bar{Y}} \right] b$$

$$+ \sum_{s=R}^{T} \mathbb{E} \left[ \mathbb{B}'(\bar{y}_R) \right] \mathbb{E} \left[ \frac{u_c(c_t, l_t) - (1 + (1 - \tau_a) r)^{s-t} \beta^{s-t} \prod_{q=1}^{s-t} \psi_{t+q} u_c(c_s)}{u_c(c_t, l_t)} \right]$$

$$- \sum_{s=R}^{T} (1 + (1 - \tau_a) r)^{s-t} \beta^{s-t} \prod_{q=1}^{s-t} \psi_{t+q} \text{Cov} \left( \mathbb{B}'(\bar{y}_R), \frac{u_c(c_s)}{u_c(c_t, l_t)} \right)$$

Finally, in order to arrive at the desired result, I once again use the identity

$$\mathbb{E} [A \cdot B] = \mathbb{E} [A] \cdot \mathbb{E} [B] + \text{Cov} (A, B)$$

to write

$$\tau^t_{\text{wedge}} = \frac{1}{R-1} \sum_{s=R}^{T} [(1 + (1 - \tau_a) r)^{s-t} - (1 + i)^{R-t}] \beta^{s-t} \prod_{q=1}^{s-t} \psi_{t+q} \mathbb{E} \left[ \mathbb{B}'(\bar{y}_R) \frac{u_c(c_s)}{u_c(c_t, l_t)} \right]$$

$$+ \sum_{s=R}^{T} \mathbb{E} \left[ b - \mathbb{B}'(\bar{y}_R) \right]$$

$$- \sum_{s=R}^{T} \left[ 1 - (1 + g)^{-(s-R)} \prod_{q=1}^{s-R+\bar{Y}} \psi_{R+q} \sum_{\bar{Y}} \right] b$$

$$+ \sum_{s=R}^{T} \mathbb{E} \left[ \mathbb{B}'(\bar{y}_R) \right] \mathbb{E} \left[ \frac{u_c(c_t, l_t) - (1 + (1 - \tau_a) r)^{s-t} \beta^{s-t} \prod_{q=1}^{s-t} \psi_{t+q} u_c(c_s)}{u_c(c_t, l_t)} \right]$$

$$- \sum_{s=R}^{T} (1 + (1 - \tau_a) r)^{s-t} \beta^{s-t} \prod_{q=1}^{s-t} \psi_{t+q} \text{Cov} \left( \mathbb{B}'(\bar{y}_R), \frac{u_c(c_s)}{u_c(c_t, l_t)} \right)$$

which completes the proof. \(\blacksquare\)
References


