Competition in defensive and offensive advertising strategies in a segmented market

Dominika Machowska*

Department of Econometrics, Faculty of Economics and Sociology, University of Łódź, Rewolucji 1905 R. No. 41, Łódź 90-214 Poland

Andrzej Nowakowski*

Faculty of Mathematics and Computer Science, University of Łódź, S. Banacha No. 22, Łódź 90-238, Poland

Abstract

We propose the new goodwill model à la Nerlove-Arrow defined on a competitive segmented market. Based on the dual dynamic approach, we give the sufficient condition under which the open-loop equilibrium exists for the new game. We also introduce $\varepsilon$-open loop equilibrium as a basis for the numerical algorithm using a construction of the optimal solution in the finite steps. The numerical algorithm enables an analysis of how the level of the homogeneity of given competitive products and customer recommendations modify optimal goodwill and the total profit of each player.

Keywords: open-loop Nash equilibrium, goodwill, market segmentation, dual dynamic approach, numerical algorithm

1. Introduction

Market segmentation is a basic tool that can increase the effectiveness of marketing activities (see Kotler et al. (2010)). Modern technologies allow marketers to adapt marketing messages to the individual preferences of each customer (see Rust and Miu (2006); Boerman et al. (2017)); for example, advertisers frequently employ targeted advertisements in the form of website

*Corresponding author

Email addresses: dominika.machowska@uni.lodz.pl (Dominika Machowska), annowako@math.uni.lodz.pl (Andrzej Nowakowski)

Preprint submitted to Elsevier May 10, 2019
banners that are specifically informed by the recent Internet searches an individual has performed (cf. Chen and Stallaert (2014)). In response to these practices, there is a requirement to create a tool that allows marketers to determine optimal advertising strategies for use in a market that is divided into infinite segments. In the existing literature, such a challenge was undertaken by Grosset and Viscolani (2005); Faggian and Grosset (2013); Górajski and Machowska (2017, 2018b). In these works, advertising strategies were investigated in a monopolistic market. In addition, competitiveness in a market of an infinite number of segments was initiated by Grosset and Viscolani (2016).

This paper further develops the idea of modelling competition in a segmented market. Namely, goodwill dynamics, defined in the infinite market segments, are included in combination with some additional factors. First, since the customer knowledge and experience in using the product is essential in reaction to advertising and in making purchasing decision (see Weilbacher (2003); Dens and De Pelsmacker (2010)) we assume that a market is divided according to customer's usage experience.

Second, managers control two types of advertising strategies over the finite decision horizon: defensive and offensive. The former is focused on retaining customers, and it is assumed that advertising strategies are tailored to the preferences and needs of consumers with different experience in using the product, so it is analysed an infinite number of defensive advertising strategies. This assumption is based on recent empirical findings that defensive marketing strategies are more effective if they are aligned with customers' needs (see Hamilton et al. (2017)). On the other hand, offensive advertising aims to attract new customers. This type of advertising was discussed recently in Martín-Herrán et al. (2012); Jørgensen and Signé (2015), however, without the assumption about the existence of market segmentation. In the new model, we postulate that companies compete for potential customers; i.e., each company invests in offensive advertising to acquire new customers.

The third new factor is customer recommendations. These are included in the new model in response to the observations made by Nielsen Research, which revealed that 78% of consumers believe in a customer recommendation rather than other sources of information about the products, and 61% perceive customer recommendations to be the most reliable source of information (see Keiningham et al. (2018)). These phenomena are particularly important for consumers who have no experience of using a given product (see East et al. (2005)). Thus, we assume that potential customers who do not possess any experience of using a product will rely on existing customers’
opinions when making a purchase decision.

As a result, we formulate a new partial differential game that allows us to determine advertising equilibrium in alignment with the individual preferences of infinite customers. The goal of each company is to maximize the total discounted profit depending on goodwill over a finite decision horizon.

In Grosset and Viscolani (2016), the necessary condition for the existence of the open-loop equilibrium for the partial differential game was proposed. In this paper, we focus on the condition under which the open-loop equilibrium exists; i.e., we formulate the sufficient condition of the existence of the open-loop equilibrium for the new game. These conditions are formulated in terms of the dual dynamic programming method, which was first introduced in Nowakowski (1992). This approach has not previously been applied to games with nonzero sum as those proposed in this paper. Two main difficulties must be overcome in such problems. The first consists of the following observation: we cannot perform perturbations of the problem—as it is considered in the fixed set with boundary condition,—which can be compared to the one-dimensional case given in Nowakowski (1992). The second difficulty is that we deal with a non-linear, non-local boundary condition with the control. The main idea of the method presented in Nowakowski (1992) is that it is carried over all objects used in dynamic programming to dual space—space of multipliers (similar to those that appear in the Pontryagin maximum principle). Next, instead of the classical value function which, in the case of games, we consider to be non-differentiable (even not continuous), we define an auxiliary function that satisfies the first order partial differential equation of dual dynamic programming. Investigations of the properties of this function lead to an appropriate verification theorem. This means that we do not need any regularity of the value function to formulate the verification theorem for the open-loop equilibrium.

However, we should bear in mind that, in general, we are not able to give the explicit formulae for the open-loop equilibrium and the associated goodwill trajectory. In the literature, we can find numerical approximations of an open-loop equilibrium, at least in the case of the linear quadratic problem. However, there are not sufficient approximate optimal criteria to state when we can stop our numerical calculation or estimate how far the value of the cost functional of the approximate equilibrium is from the exact optimal value of the cost (which is not known). In most papers, only convergence of numerical approximation is stated and, in fact, only for its subsequence. Using any known numerical method, in practice, we calculate an approximate sequence. However, we do not known whether it is only the sequence that is convergent; we only know that it contains a convergent subsequence.
Therefore, we require the sufficient optimality criteria for the approximate equilibrium. The second aim of this paper is to describe the construction of a dual dynamic programming method that finalises the verification theorem to prove the existence of an approximate equilibrium, a so-called \( \varepsilon \)-open-loop equilibrium. The main idea is to show that the difference between the value of the goal functional and an \( \varepsilon \)-open-loop equilibrium is less than the given \( \varepsilon \).

The numerical algorithm is used to characterize the optimal goodwill under different degrees of the product homogeneity and different levels of own and competitor’s customer recommendation. Therefore, we can answer the following questions: (i) Do the equilibrium advertising strategies always increase goodwill relative to its initial level? (ii) Who wins or loses less when competitive products are similar? (iii) By how much does the customer recommendation increases the total profit?

The rest of the paper is organized as follows. Section 2 contains an introduction to the new goodwill model, which generates a partial differential game. Section 3 contains the formulation for the dual optimal game and the verification theorem. Section 4 presents a formulation of the approximate verification theorem, which represents a base from which we can propose the numerical algorithm to search for the \( \varepsilon \)-open-loop equilibrium. Section 5 presents the results of simulations of the advertising equilibrium and associated goodwill obtained by the proposed algorithm. Finally, the conclusions are presented in Section 6.

2. Model Formulation

Let us consider two competitive companies operating in the same segmented market. The market is divided according to customers’ usage experiences which are normalized to 1; i.e., \( a \in [0, 1] \) similar to Górajski and Machowska (2018b). In segment \( a = 0 \), there are potential customers who do not have any experience of using the product; however, they declare sufficient interest in buying it. Segment \( a = 1 \) consists of customers who stop buying and using the product permanently. We assume that the decision horizon is finite \( T \in (0, \infty) \). In each time, \( t \in [0, T] \), and each market segment, \( a \), both companies invest in defensive advertising efforts, \( u_i(t, a) \). Thus, managers control infinite personalized advertising tools through which they can build player’s \( i \) goodwill, \( x_i(t, a) \), which is monitored in each unit time, \( t \), and each market segment, \( a \). As in Nerlove and Arrow (1962) and Grosset and Viscolani (2008), we define goodwill as the part of demand that
is created by advertising efforts. Therefore, defensive strategies positively influence goodwill among customers with some experiences and the evolution of goodwill depends only on its own effort. This reflects a hypothesis that among customers with some experience competitors’ advertising has only negligible impact on goodwill in these segments (see Stammerjohan et al. (2005)). However, the existence of a competitor gives the opportunity to choose different product if the one that was previously purchased does not meet the customer’s expectations. Nevertheless, such decisions are made on the basis of their own preferences and experience and not influenced by the competitor’s actions. Therefore, we assume that the exogenous depreciation rate of goodwill $\delta_i(a) > 0$ for $a \in [0, 1]$ takes into account the phenomenon of consumer departure to a competitor $\delta_{i,3-i}(a)$, moving from a competitor, $\delta_{3-i,i}(a)$ and leaving the market permanently, $\tilde{\delta}_i(a)$, thus

$$\delta_i(a) = \delta_{i,3-i}(a) - \delta_{3-i,i}(a) + \tilde{\delta}_i(a). \quad (1)$$

Moreover, we assume that, as time passes, each customer gains more and more experience; thus, she progresses to the next market segment. Therefore, the goodwill dynamics for player $i$, $i = 1, 2$ is given by

$$\frac{\partial x_i(t,a)}{\partial t} = -\frac{\partial x_i(t,a)}{\partial a} - \delta_i(a)x_i(t,a) + \lambda_{u,i}(a)u_i(t,a) \quad (2)$$

for $(t,a) \in [0,T] \times [0,1]$ and $\lambda_{u,i}(a)$ reflects the effectiveness of defensive strategies of player’s $i$ directed to a segment $a$. Equation (2) states that the rate of change of goodwill, $\frac{\partial x_i(t,a)}{\partial t}$ is decreasing by the flow of goodwill from $\frac{\partial x_i(t,a)}{\partial a}$ from one market segment to another and by the depreciation rate $\delta_i(a)x_i(t,a)$ and in the same time, goodwill is enhanced by action of defensive advertising $\lambda_{u,i}(a)u_i(t,a)$.

We assume that, at the beginning of the game, each company has a certain level of the goodwill as a result of pre-launch strategies (see Buratto and Viscolani (2002); Górajski and Machowska (2018b)). Thus, at the initial time, $t = 0$, we have

$$x_i(0,a) = x_{i,0}(a) > 0, \quad a \in [0,1]. \quad (3)$$

Next, we define goodwill in the new customer segment. We assume that each company has its own level of customer recommendations, denoted by $\tilde{R}_i(a)$, that differ among consumers with varying degrees of experience in the use of the product. Each company acquires new consumers when it is positively evaluated by existing consumers; i.e., when recommendations related to the
product are higher than the recommendations of the competitor’s product. However, we assume that the judgement of the customers who are already using the product that a potential customer intends to purchase is more important than their opinion about a competitor’s product. This inequality is measured by $\gamma_i^{R_{3-i}}$. Since we focus only on the positive recommendations, we define them in each market segment $a \in [0, 1]$ as follows:

$$R_i(a) = \begin{cases} \tilde{R}_i(a) - \gamma_i^{R_{3-i}} \bar{R}_{3-i}(a) & \text{if } \tilde{R}_i(a) - \gamma_i^{R_{3-i}} \bar{R}_{3-i}(a) > 0 \\ 0 & \text{otherwise} \end{cases}.$$  

Next, we also consider an offensive advertising denoted by $v_i(t)$. This type of marketing strategies focuses on acquiring new customers. We assume that there is a competition between both firms for the potential customers; thus, for each player $i$ we define for $t \in [0, T]$

$$x_i(t, 0) = \int_0^1 R_i(a)x_i(t, a)da + \lambda_{v,i}v_i(t) - \gamma_{i,3-i}\lambda_{v,3-i}v_{3-i}(t).$$  

The parameter $\gamma_{i,3-i}$ reflects the impact of the competitor’s advertising on goodwill among potential consumers and varies according to product homogeneity. We assume, like Amrouche et al. (2008), that the marginal impact of a brand’s own advertising on the goodwill is greater than the marginal impact of the competitor’s advertising; therefore, $\gamma_{i,3-i} \in (0, 1)$. As a result, the greater the differentiation between products, the lower the value $\gamma_{i,3-i}$.

Next, we formulate the goal of each player. Let us consider the demand function, $q_i(t, a)$, in the time, $t$, and in the market segment, $a$, (similar to Nerlove and Arrow (1962)):

$$q_i(t, a) = z_i(t, a)x_i(t, a),$$  

where the function, $z_i$, reflects all exogenous aspects that may have an impact on demand such as the price of a product or customer incomes. Considering the linear operating costs of the form $\tilde{C}(q_i) = \nu_iq_i + c_f$ for the unit costs, given by $\nu_i$, and fixed cost, indicated by $c_f$, we define the instantaneous profit for each segment $a$

$$\Pi_i(t, a) = p_i(t, a)q_i(t, a) - \tilde{C}(q_i)(t, a) = \pi_i(t, a)x_i(t, a) - c_f$$  

for $\pi_i(t, a) = z_i(t, a)\left(p_i(t, a) - c_f\right)$ and $p_i(t, a)$ is a product’s price at time $t$ in segment $a$. Moreover, taking as a base the broad literature related to dynamic advertising competition, summarized in Huang et al. (2012), we
state that the advertising efforts of each company entail a quadratic cost of the form

\[ C_A(u_i) = \frac{\beta_{u,i}}{2} u_i^2, \quad C_A(v_i) = \frac{\beta_{v,i}}{2} v_i^2, \]

where \( \beta_{u,i} > 0 \) and \( \beta_{v,i} > 0 \) are the unit costs of defensive and offensive advertising, respectively, for the player \( i \).

The goal of each player is to maximize the total discounted profit given by

\[ J_i(u_i, v_i, v_{3-i}) = \int_0^T e^{-r_i t} \int_0^1 \left[ \Pi_i(t, a) - C_A(u_i(t, a)) \right] da dt \]

\[ - \int_0^T e^{-r_i t} C_A(v_i(t)) dt \]

\[ = \int_0^T e^{-r_i t} \int_0^1 \left( \pi_i(t, a) x_i(t, a) - \frac{\beta_{u,i}}{2} u_i^2(t, a) \right) da dt \]

\[ - \int_0^T e^{-r_i t} \frac{\beta_{v,i}}{2} v_i^2(t) dt, \]

(5)

where \( r_i > 0 \) is a discount rate employed by firm \( i \).

As a result, the model (2) - (4) with (5) constructs the partial differential game (henceforth referred to as \((G)\)). In the next section, we investigate a sufficient condition for an existence of the open-loop Nash equilibrium for the new game.

3. Nash Equilibrium

For the player \( i, i = 1, 2 \) the set of admissible controls is given by

\[ U_i = \{(u_i, v_i, v_{3-i}): u_i: [0, T] \times [0, 1] \to [0, K] \text{ is measurable} \]

and for each \( i = 1, 2 \) \( v_i: [0, T] \to [0, K] \) is measurable \}

for the maximal advertising intensity \( K > 0 \). Trajectory \( x_i: [0, T] \times [0, 1] \to \mathbb{R} \) satisfies (2-4) for \( (u_i, v_i, v_{3-i}) \in U_i \) is called admissible state. For each player \( i, i = 1, 2 \), the set of all admissible controls and states is denoted by \( Ad_i \).

The goal for this section is to investigate the open-loop Nash equilibrium for game \((G)\) defined as
Definition 1. An admissible triple $\left( \bar{u}_i, \bar{v}_i, \bar{v}_{3-i} \right) \in U_i$, $i = 1, 2$ for the partial differential game $(G)$ is the open-loop Nash equilibrium if and only if for $i = 1, 2$ and for all admissible controls $(u_i, v_i, v_{3-i}) \in U_i$ the following inequalities hold

$$J_i(\bar{u}_i, \bar{v}_i, \bar{v}_{3-i}) \geq J_i(u_i, v_i, v_{3-i}).$$

The trajectory $x_i$ associated with the triple $(\bar{u}_i, \bar{v}_i, \bar{v}_{3-i})$ we will call as the optimal goodwill.

As we mention in the Introduction, the game $(G)$ depends on time, we cannot perturb it with respect to the initial data (which are fixed as in $[3]$) and the time (nonlocal condition) as it is usually done in the classical optimal control problem, (cf. Cesari (2012)), in order to prove the sufficient condition of the existence of the equilibrium. This is why a dual dynamic approach to the above problem seems to be the only one that suits. In the next section, we introduce the dual game for $(G)$.

### 3.1. Dual optimal game for $(G)$

Denote the set $Q = \{(t, a): 0 < t < T, 0 < a < 1\}$. For the player $i$, $i = 1, 2$ let $P_i \subset \mathbb{R}^{2+2}$ be an open set of the form

$$P_i = \{(t, a, p_i) = (t, a, y_i^0, y_i), y_i^0 < 0, y_i \in \mathbb{R}, (t, a) \in Q\}.$$

For $i = 1, 2$ we denote by $clP_i$ the closure of $P_i$ and the projection of $P_i$ by $P_i^0 = \{(t, p_i): (t, 0, p_i) \in P_i\}$. Now, let us introduce auxiliary functions $V^i : clP_i \rightarrow \mathbb{R}$ for $i = 1, 2$. We assume that these functions belong to $H^2(P_i)$ (Sobolev space of functions having second weak derivatives) and satisfy so-called the transversality condition:

$$V^i(t, a, p_i) = y_i^0 V^i_{y_i^0}(t, a, p_i) + y_i V^i_{y_i}(t, a, p_i), \quad (t, a, p_i) \in clP_i, \quad i = 1, 2,$$  \hspace{1cm} (6)

where $V^i_{y_i^0}, V^i_{y_i}$ express partial derivatives of $V^i$. Functions $x_i : clP_i \rightarrow \mathbb{R}$, $i = 1, 2$ are defined by

$$x_i(t, a, p_i) = -V^i_{y_i}(t, a, p_i), \quad (t, a, p_i) \in clP_i, \quad i = 1, 2.$$  \hspace{1cm} (7)

In the further part of the paper we use the set

$$Ad_{X_i} = \{(u_i, v_i, v_{3-i}, x_i) \in Ad_i: \exists p_i(t, a) = (y_i^0, y_i(t, a)), (t, a) \in Q,$$

such that $y_i^0 < 0, y_i \in H^2(Q)$, $(t, a, p_i(t, a)) \in clP_i, \ y_i(T, a) = 0, \ y_i(t, 1) = 0, \ x_i(t, a) = x_i(t, a, p_i(t, a)), \ (t, a) \in clQ\}.$
Henceforth, we assume that $Ad_{x_i} \neq \emptyset$ for $i = 1, 2$.

The function $p_i : Q \to \mathbb{R}^2$ we call the **dual trajectory**, while $x_i : Q \to \mathbb{R}$ we call the **primal trajectory** for the player $i$. The function $x_i$ builds a relationship between a dual and a primal trajectory which is defined in the set $Ad_{x_i}$.

To find the open-loop Nash equilibrium we formulate the dual Hamilton-Jacobi equations for $i = 1, 2$ in $P_i$ as

$$
\max_{u_i \in [0,K]} \left\{ V_i^j(t,a,p_i) + V_a^j(t,a,p_i) + y_i \left[ \delta_i(a)V_{y_i}^j(t,a,p_i) + \lambda_{u,i}(a)u_i \right] \right\} - y_i^0 e^{-\tau_i t} \left[ \pi_i(t,a) \left( -V_{y_i}^j(t,a,p_i) \right) - \frac{\beta_{v,i}}{2} u_i^2 \right] = 0
$$

(8)

with the initial conditions

$$
\int_0^1 V_{y_i}^j(0,a,p_i) da = 0, \quad (0,a,p_i) \in P_i.
$$

Moreover, the dual boundary Hamilton-Jacobi type equations defined on $P_{i}^0$ for $i = 1, 2$ is defined as

$$
-V_i^j(t,0,p_i) = \max_{v_i,v_{3-i} \in [0,K]} \left\{ y_i^0 e^{-\tau_i t} \frac{\beta_{v,i}}{2} v_i^2 
+ y_i \left( \int_0^1 R_i(a)(-V_i^j(t,a,p_i)) da + \lambda_{v,i}v_i - \gamma_{v,3-i}v_{3-i} \right) \right\}
$$

(10)

with the conditions

$$
\int_0^T V_{y_i}^j(t,0,p_i) dt = \alpha_i \quad (t,0,p_i) \in P_i^0,
$$

(11)

where $\alpha_i$ is a positive constant.

### 3.2. The verification theorem

The dual approach to the dynamic programming described in the former section allows us to formulate and to prove a kind of a verification theorem ensuring sufficient optimality conditions for the new game ($G$). We would like to stress that we work now in the dual space $clP_i$ and with the auxiliary function $V^i$ defining the set $Ad_{x_i}$. We should have in mind that if we change the functions $V^i$, then we also change the set $Ad_{x_i}$. Let us fix $y_i^0 < 0$. Define the set $P_i, i = 1, 2$, which is also determined by $V^i$

$$
P_i = \left\{ p_i(t,a) = (y_i^0, y_i(t,a)), \quad (t,a,p_i(t,a)) \in clP_i, \quad y_i \in H^2(Q), \quad y_i(T,a) = 0, \quad y_i(t,1) = 0, \quad (u_i,v_i,v_{3-i},x_i) \in Ad_{x_i}, \quad x_i(t,a) = -V_{y_i}^j(t,a,p_i(t,a)), \quad (t,a) \in clQ \right\}.
$$
Theorem 2. Assume that there exist \( V^i \in H^2(P_i), \ i = 1, 2 \) satisfying (6) and (11). Take \((\bar{y}^i_0, \bar{y}^i(t, a)) = \bar{p}_i(t, a) \in \mathcal{P}^i\) and \((\bar{u}_i, v_i, \bar{v}_{3-i}, \bar{x}_i) \in \text{Ad}_{x_i}, i = 1, 2\) such that for \((t, a) \in \text{cl}Q,\) we get \( \bar{x}_i(t, a) = -V^i_{y_i}(t, a, \bar{p}_i(t, a)) \) and the equalities are satisfied

\[
V^i(t, a, \bar{p}_i(t, a)) + V^i_{a}(t, a, \bar{p}_i(t, a)) + \bar{y}^i(t, a) \left( \delta_i(a)V^i_{y_i}(t, a, \bar{p}_i(t, a)) + \lambda_{u,i}(a)\bar{u}_i(t, a) \right)
- \bar{y}^0_ie^{-r_it} \left[ -\pi_i(t, a)V^i_{y_i}(t, a, \bar{p}_i(t, a)) - \frac{\beta_{u,i}}{2}\bar{u}_i^2(t, a) \right] = 0 \tag{12}
\]

and

\[
- V^i(t, 0, \bar{p}_i(t, 0)) = \bar{y}^0_0 e^{-r_i0} \frac{\beta_{v,i}}{2}\bar{v}_i^2(t)
+ \bar{y}^i_{1}(t, 0) \int_0^1 R_i(a) \left( -V^i_{y_i}(t, a, \bar{p}_i(t, a)) \right) \, da 
+ \bar{y}^i_{1}(t, 0) \left( \lambda_{v_i}\bar{v}_i(t) - \gamma_{i,3-i}\lambda_{v,3-i}\bar{v}_{3-i}(t) \right) \tag{13}
\]
for \((t, 0, \bar{p}_i(t, 0)) \in \mathcal{P}^i_0.\) Then \((\bar{u}_i, v_i, \bar{v}_{3-i})\) is the open loop Nash equilibrium and \(\bar{x}_i\) is the optimal trajectory associated with \((\bar{u}_i, v_i, \bar{v}_{3-i}).\)

Proof. Fix \(i\) and take any \((u_i, v_i, \bar{v}_{3-i}, x_i) \in \text{Ad}_{x_i}\) and corresponding \(p_i \in \mathcal{P}^i\) such that \(x_i(t, a) = -V^i_{y_i}(t, a, p_i(t, a)),\) \((t, a) \in Q.\) From transversality condition (6) we infer that for \((t, a, p_i(t, a)) \in P_i,\)

\[
V^i(t, a, p_i(t, a)) + V^i_{a}(t, a, p_i(t, a)) =
\bar{y}^i_0 \left( \frac{d}{dt}V^i_{y_i}(t, a, p_i(t, a)) + \frac{d}{da}V^i_{y_i}(t, a, p_i(t, a)) \right)
+ y_i(t, a) \left( \frac{d}{dt}V^i_{y_i}(t, a, p_i(t, a)) + \frac{d}{da}V^i_{y_i}(t, a, p_i(t, a)) \right). \tag{14}
\]

From (2)-(3), we have

\[
\frac{d}{dt}V^i_{y_i}(t, a, p_i(t, a)) + \frac{d}{da}V^i_{y_i}(t, a, p_i(t, a)) = 
- \delta_i(a)V^i_{y_i}(t, a, p_i(t, a)) - \lambda_{u,i}(a)u_i(t, a). \tag{15}
\]

Putting (15) into (14) and applying (8), we get the inequality

\[
\bar{y}^0_0 \left( \frac{d}{dt}V^i_{y_i}(t, a, p_i(t, a)) + \frac{d}{da}V^i_{y_i}(t, a, p_i(t, a)) \right)
\leq \bar{y}^0_0 e^{-r_it} \left[ -\pi_i(t, a)V^i_{y_i}(t, a, p_i(t, a)) - \frac{\beta_{u,i}}{2}u_i^2(t, a) \right]. \tag{16}
\]
Next, we integrate (16) on $cI_Q$, thus we get

$$
\tilde{y}_i^0 \int_0^1 V_{y_i}^i(T, a, p_i(T, a)) da - \tilde{y}_i^0 \int_0^1 V_{y_i}^i(0, a, p_i(0, a)) da
+ \tilde{y}_i^0 \int_0^T (V_{y_i}^i(t, 1, p_i(t, 1)) - V_{y_i}^i(t, 0, p_i(t, 0))) dt
\leq \tilde{y}_i^0 \int_0^T e^{-r_i t} \left[ \int_0^1 \left( -\pi_i(t, a)V_{y_i}^i(t, a, p_i(t, a)) - \frac{\beta_{u,i}}{2} u_i^2(t, a) \right) da \right] dt, \quad (17)
$$

Using the conditions (9) and (11), (17) is reduced to

$$
\tilde{y}_i^0 \int_0^1 V_{y_i}^i(T, a, p_i(T, a)) da + \tilde{y}_i^0 \int_0^T V_{y_i}^i(t, 1, p_i(t, 1)) dt
\leq \tilde{y}_i^0 \int_0^T e^{-r_i t} \left[ \int_0^1 \left( -\pi_i(t, a)V_{y_i}^i(t, a, p_i(t, a)) - \frac{\beta_{u,i}}{2} u_i^2(t, a) \right) da \right] dt. \quad (18)
$$

Following the same way but now using (12), we come to the equality

$$
\tilde{y}_i^0 \int_0^1 V_{y_i}^i(T, a, \bar{p}_i(T, a)) da + \tilde{y}_i^0 \int_0^T V_{y_i}^i(t, 1, \bar{p}_i(t, 1)) dt
= \tilde{y}_i^0 \int_0^T e^{-r_i t} \left[ \int_0^1 \left( -\pi_i(t, a)V_{y_i}^i(t, a, \bar{p}_i(t, a)) - \frac{\beta_{u,i}}{2} u_i^2(t, a) \right) da \right] dt. \quad (19)
$$

Next, applying the transversality condition to $V_i(t, 0, p_i(t, 0))$ at the points belonging to $P_i^0$, we have

$$
V_i(t, 0, p_i(t, 0)) = \tilde{y}_i^0 V_{y_i}^i(t, 0, p_i(t, 0)) + y_i(t, 0)V_{y_i}^i(t, 0, p_i(t, 0)). \quad (20)
$$

Moreover, by (11), we have in $P_i^0$

$$
- V_{y_i}^i(t, 0, p_i(t, 0)) = \int_0^1 R_i(a)(-V_{y_i}^i(t, a, p_i(t, a))) da + \lambda_{v,i}v_i(t) - \gamma_{i,3-i}v_{3-i}(t). \quad (21)
$$

Finally, putting (21) into (20) and using the dual Hamilton-Jacobi type equation on $P_i^0$ defined by (10) we get for any admissible control $v_i$ the inequality at $P_i^0$

$$
V_i(t, 0, p_i(t, 0)) = \tilde{y}_i^0 V_{y_i}^i(t, 0, p_i(t, 0))
- y_i(t, 0) \left( \int_0^1 R_i(a)(-V_{y_i}^i(t, a, p_i(t, a))) da + \lambda_{v,i}v_i(t) - \gamma_{i,3-i}v_{3-i}(t) \right)
\geq \tilde{y}_i^0 V_{y_i}^i(t, 0, p_i(t, 0)) + \tilde{y}_i^0 e^{-r_i t v_i^2(t)} + V_i(t, 0, p_i(t, 0)). \quad (22)
$$
As a result we get
\[ \bar{y}_1^0 V_i^1(t, 0, p_i(t, 0)) \leq -\bar{y}_1^0 e^{-r_1 t} \bar{v}_1^2(t). \] (23)

Using the same arguments, we get the equality at \( P_i^0 \)
\[ \bar{y}_i^0 V_i^1(t, 0, \bar{p}_i(t, 0)) = -\bar{y}_i^0 e^{-r_i t} \bar{v}_i^2(t). \] (24)

Finally, by (18), (19) and the fact that \( \bar{p}_i(T, a) = p_i(T, a) = 0, \bar{p}_i(t, 1) = p_i(t, 1) = 0 \) we get
\[ \bar{y}_i^0 (J(\bar{u}_i, \bar{v}_i, \bar{v}_{3-i}) - J(u_i, v_i, \bar{v}_{3-i})) \]
\[ = \bar{y}_i^0 \int_0^T e^{-r_i t} \int_0^1 \left( \pi_i(t, a) \left(-V_i(t, a, \bar{p}_i(t, a))\right) - \frac{\beta_{u,i}}{2} \bar{u}_i^2(t, a) \right) dadt \]
\[ - \bar{y}_i^0 \int_0^T e^{-r_i t} \int_0^1 \left( \pi_i(t, a) \left(-V_i(t, a, p_i(t, a))\right) - \frac{\beta_{u,i}}{2} u_i^2(t, a) \right) dadt \]
\[ - \bar{y}_i^0 \int_0^T e^{-r_i t} \frac{\beta_{u,i}}{2} (\bar{v}_i^2(t) - v_i^2(t)) dt \]
\[ \leq -\bar{y}_i^0 \int_0^T e^{-r_i t} \frac{\beta_{u,i}}{2} (\bar{v}_i^2(t) - v_i^2(t)) dt \leq 0. \] (25)

The last inequality is obtained from (23) and (24) and the proof is complete.

4. Approximate Optimality in Game (G)

In the above subsections we described the optimality conditions for problem (G) in terms of the dual dynamic programming. In practice, it is difficult to find solutions in the exact form as stated there. In this section, we use the same notations as in the former sections and we present the dual dynamic approach to sufficient conditions for an approximate (\( \varepsilon \)-optimality) optimality. To this effect for given \( \bar{v}_{3-i} \) and fixed \( \bar{y}_{\varepsilon,i}^0 < 0 \) define a dual optimal values \( S_D^i, i = 1, 2, \)
\[ S_D^i = \max_{(u_i, v_i, \bar{v}_{3-i}, x_i) \in A_{x_i}} -\bar{y}_{\varepsilon,i}^0 J_i(u_i, v_i, \bar{v}_{3-i}). \] (26)

Next, for each player \( i, i = 1, 2, \varepsilon > 0 \) and \( x_i, \) we define a dual \( \varepsilon \)-optimal value for (G) as \( S_{\varepsilon,D}^i \) which satisfies the condition
\[ S_D^i \leq S_{\varepsilon,D}^i \leq S_D^i - \varepsilon \bar{y}_{\varepsilon,i}^0, \] (27)
where $S^i_D$ is defined by (26). The condition (27) means (assuming $\gamma^0_{\varepsilon,i} = -1$) that we are looking for such admissible controls $\bar{u}_{\varepsilon,i}, \bar{v}_{\varepsilon,i}, \bar{v}_{\varepsilon,3-i}$ which will lead the state $\bar{x}_{\varepsilon,i}$ to give $J_i(\bar{u}_{\varepsilon,i}, \bar{v}_{\varepsilon,i}, \bar{v}_{\varepsilon,3-i})$ such that
\[
J_i(\bar{u}_{\varepsilon,i}, \bar{v}_{\varepsilon,i}, \bar{v}_{\varepsilon,3-i}) \geq J_i(u_i, v_i, \bar{v}_{3-i}) - \varepsilon \tag{28}
\]
for all $(u_i, v_i, \bar{v}_{3-i}, x_i) \in Ad_{X_i}$.

Let us consider the dual Hamilton-Jacobi inequality in $P_i$ for $i = 1, 2$
\[-\varepsilon G^{i,0}_{\varepsilon,i} \geq \max_{u_i \in [0,K]} \left\{ \bar{V}_t^i(t, a, p_i) + \bar{V}_a^i(t, a, p_i) + y_i \left( \delta_i(a) \bar{V}_y^i(t, a, p_i) + \lambda_{u,i} u_i \right) - \frac{\beta_{u,i}}{2} \bar{u}_i^2 \right\} \geq 0 \tag{29}
\]
with the initial conditions
\[
\int_0^1 \bar{V}_y^{i,0}(0, a, p_i) da = 0, \quad (0, a, p_i) \in P_i. \tag{30}
\]
Moreover, the boundary dual Hamilton-Jacobi inequality on $P_i^0$ for $i = 1, 2$ takes a form
\[-\varepsilon G^{i,0}_{\varepsilon,i} \geq \bar{V}_t^i(t, 0, p_i) + \max_{v_i, v_{3-i} \in [0,K]} \left\{ \bar{V}_t^{i,0} e^{-r_i t} \frac{\beta_{v,i}}{2} \bar{v}_i^2 \right\} + \lambda_{v,i} v_i - (\gamma_{i,3-i} + \lambda_{v,3-i} v_{3-i}) \right\} \geq 0 \tag{31}
\]
with the conditions
\[
\int_0^T \bar{V}_y^{i,0}(t, 0, p_i) dt = \alpha_i, \quad (t, 0, p_i) \in P_i^0, \tag{32}
\]
where $\alpha_i$ is a positive constant. Therefore, the system of inequalities for the auxiliary functions $\bar{V}_i$ is defined by (29)-(32). Since our goal is to apply the approximate dual approach to numerical solutions of (2)-(4), instead of the system of equations, for fixed $m_i > 0$ we consider the inequalities:
\[
0 \leq \frac{\partial x_{\varepsilon,i}(t, a)}{\partial t} + \frac{\partial x_{\varepsilon,i}(t, a)}{\partial a} + \delta_i(a) x_{\varepsilon,i}(t, a) - \lambda_{u,i} u_{\varepsilon,i}(t, a) - \frac{\varepsilon}{m_i} G^{i,0}_{\varepsilon,i} \tag{33}
\]
for $(t, a) \in clQ$ which satisfy the boundary conditions for $t \in [0, T]$
\[
0 \leq x_{\varepsilon,i}(t, 0) - \int_0^1 R_i(a) x_{\varepsilon,i}(t, a) da - \lambda_{v,i} v_{\varepsilon,i}(t) + \gamma_{i,3-i} \lambda_{v,3-i} v_{\varepsilon,3-i}(t) \leq \frac{\varepsilon}{m_i} G^{i,0}_{\varepsilon,i} \tag{34}
\]
and the initial conditions for } \ a \in [0, 1]  \\

x_{\varepsilon,i}(0,a) = x_{\varepsilon,i,0}(a). \tag{35} \\

Thus, in this section by the set of admissible controls and states; i.e., satisfying [33]-[35], we denote by } \ Ad_{\varepsilon,i}.  \\

4.1. The approximate dual approach to the game (G)  \\

Now, we describe the concept of the } \ \varepsilon\text{-open loop Nash equilibrium } \ (\bar{u}_{\varepsilon,i}, \bar{v}_{\varepsilon,i}, \bar{v}_{\varepsilon,3-i}) \text{ and prove a sufficient } \varepsilon\text{-optimality for game (G) i.e. an } \varepsilon\text{-version of the verification theorem. Assume that there exist } \tilde{V}^i \text{ satisfying [6] and [29]-[32]. Then we define}  \\

x_{\varepsilon,i}(t,a,p_i) = -\tilde{V}^i_{y_{\varepsilon}}(t,a,p_i), (t,a,p_i) \in clP_1. \tag{36} \\

As well, for fixed } \varepsilon > 0, \varepsilon > 0 \text{ we characterise}  \\

Ad_{x_{\varepsilon,i}} = \{ (u_{\varepsilon,i}, v_{\varepsilon,i}, x_{\varepsilon,i}) \in Ad_{\varepsilon,i} : \exists p_{\varepsilon,i}(t,a) = (y_{\varepsilon,i}^0, y_{\varepsilon,i}(t,a)), (t,a) \in Q, \text{ such that } \ y_{\varepsilon,i} \in H^2(Q), \ y_{\varepsilon,i}(T,a) = 0, \ y_{\varepsilon,i}(t,1) = 0, \ x_{\varepsilon,i}(t,a) = x_{\varepsilon,i}(t,a, p_{\varepsilon,i}(t,a)), (t,a) \in clQ \}  \\

and for fixed } m_i > 0 \text{ we determine}  \\

P_{\varepsilon} = \{ p_{\varepsilon,i}(t,a) = (y_{\varepsilon,i}^0, y_{\varepsilon,i}(t,a)) : (t,a, p_{\varepsilon,i}(t,a)) \in clP_1, \ y_{\varepsilon,i} \in H^2(Q), \sup_{(t,a) \in Q} |y_{\varepsilon,i}(t,a)| \leq m_i, \ y_{\varepsilon,i}(t,a) > 0, \ y_{\varepsilon,i}(T,a) = 0, \ y_{\varepsilon,i}(t,1) = 0, \exists (u_{\varepsilon,i}, v_{\varepsilon,i}, v_{\varepsilon,3-i}, x_{\varepsilon,i}) \in Ad_{x_{\varepsilon,i}}, \ x_{\varepsilon,i}(t,a) = -\tilde{V}^i_{y_{\varepsilon}}(t,a, p_{\varepsilon,i}(t,a)), (t,a) \in clQ \}.  \\

Next, we propose the notion of } \varepsilon\text{-open loop Nash equilibrium } \ (\bar{u}_{\varepsilon,i}, \bar{v}_{\varepsilon,i}, \bar{v}_{\varepsilon,3-i}).  \\

Definition 3. For given } \varepsilon > 0 \text{ and } \tilde{V}^i \in H^2(P_1) \text{ satisfying [6], [29]-[32], let } x_{\varepsilon,i}(t,a,p_i) \text{ in } P_1 \text{ be defined by [36]. Let } p_{\varepsilon,i} \in P_{\varepsilon} \text{ be defined in } Q \text{ and } \bar{x}_{\varepsilon,i}(t,a) = x_{\varepsilon,i}(t,a, p_{\varepsilon,i}(t,a)) \text{ for } (t,a) \in Q. \text{ Let } (\bar{u}_{\varepsilon,i}, \bar{v}_{\varepsilon,i}, \bar{v}_{\varepsilon,3-i}) \text{ be any admissible triple of controls such that } (\bar{u}_{\varepsilon,i}, \bar{v}_{\varepsilon,i}, \bar{v}_{\varepsilon,3-i}, x_{\varepsilon,i}) \in Ad_{x_{\varepsilon,i}}. \text{ We call the triple } (\bar{u}_{\varepsilon,i}, \bar{v}_{\varepsilon,i}, \bar{v}_{\varepsilon,3-i}) \text{ as the } \varepsilon\text{-open loop Nash equilibrium with respect to all admissible } (u_{\varepsilon,i}, v_{\varepsilon,i}, v_{\varepsilon,3-i}, x_{\varepsilon,i}) \in Ad_{x_{\varepsilon,i}} \text{ if}  \\

-\bar{y}^0_{\varepsilon,i}J_i(\bar{u}_{\varepsilon,i}, \bar{v}_{\varepsilon,i}, \bar{v}_{\varepsilon,3-i}) \geq -\bar{y}^0_{\varepsilon,i}J_i(u_{\varepsilon,i}, v_{\varepsilon,i}, v_{\varepsilon,3-i}) - \varepsilon. \tag{37}  \\

Trajectory } \bar{x}_{\varepsilon,i} \text{ is called as an } \varepsilon\text{-optimal trajectory associated with the triple } (\bar{u}_{\varepsilon,i}, \bar{v}_{\varepsilon,i}, \bar{v}_{\varepsilon,3-i}).
Now we are in a position to formulate and to prove the verification theorem for $\varepsilon$-open-loop equilibrium.

**Theorem 4.** Assume that there exist $\tilde{V}^i \in H^2(P_i)$ satisfying (6) and (29)-(32). Take $\bar{p}_{\varepsilon,i}(t,a) \in P^i_{\varepsilon}$ and $(\bar{u}_{\varepsilon,i}, \bar{v}_{\varepsilon,i}, \bar{v}_{\varepsilon,3-i}, x_{\varepsilon,i}) \in Ad_{X_i,\varepsilon}$ such that $x_{\varepsilon,i}(t,a) = -\tilde{V}^i_{y_i}(t,a,\bar{p}_{\varepsilon,i}(t,a))$ for $(t,a) \in clQ$. Moreover, assume that the $(\bar{u}_{\varepsilon,i}, \bar{v}_{\varepsilon,i}, \bar{v}_{\varepsilon,3-i}, \bar{p}_{\varepsilon,i})$ satisfies

$$-\varepsilon y^0_{\varepsilon,i} \leq \tilde{V}^i(t,a,\bar{p}_{\varepsilon,i}(t,a)) + \bar{V}^i_a(t,a,\bar{p}_{\varepsilon,i}(t,a)) + \bar{y}_{\varepsilon,i}(t,a)(\delta_i(a)\tilde{V}^i_{y_i}(t,a,\bar{p}_{\varepsilon,i}(t,a)) + \lambda_{u,i}\bar{u}_{\varepsilon,i}(t,a)) - \frac{\beta_{u,i}}{2}\bar{u}_{\varepsilon,i}^2(t,a)$$

(38)

for $(t,a) \in Q$ and $i = 1, 2$; as well as,

$$\varepsilon y^0_{\varepsilon,i} \geq \tilde{V}^i(t,0,\bar{p}_{\varepsilon,i}(0)) + \bar{y}^0_{\varepsilon,i} e^{-r_i t} \frac{\beta_{u,i}}{2}\bar{v}_{\varepsilon,i}^2(t) + \bar{y}_{\varepsilon,i}(t,0) \int_0^t R_i(a)(-\tilde{V}^i_{y_i}(t,a,\bar{p}_{\varepsilon,i}(t,a))da$$

$$+ \bar{y}_{\varepsilon,i}(t,0)(\lambda_{v,i}\bar{v}_{\varepsilon,i}(t) - \gamma_{i,3-i}\lambda_{v,i}\bar{v}_{\varepsilon,3-i}(t))$$

(39)

for $(t,0,\bar{p}_{\varepsilon,i}(t,0)) \in P^0_i$. Then $(\bar{u}_{\varepsilon,i}, \bar{v}_{\varepsilon,i}, \bar{v}_{\varepsilon,3-i})$ is the $\varepsilon$-open loop Nash equilibrium and $x_{\varepsilon,i}$ is an $\varepsilon$-optimal trajectory associated with the triple $(\bar{u}_{\varepsilon,i}, \bar{v}_{\varepsilon,i}, \bar{v}_{\varepsilon,3-i})$.

**Proof.** Fix $i$ and take any $(u_{\varepsilon,i}, v_{\varepsilon,i}, \bar{v}_{\varepsilon,3-i}, x_{\varepsilon,i}) \in Ad_{X_i,\varepsilon}$ and $p_{\varepsilon,i} \in P^i_{\varepsilon}$ such that $x_{\varepsilon,i}(t,a) = -\tilde{V}^i_{y_i}(t,a,p_{\varepsilon,i}(t,a))$, $(t,a) \in clQ$. The main idea of the proof
is the same as in the proof of Theorem 2, i.e.,

\[
\bar{y}_{\varepsilon,i}^0 \int_0^T e^{-r_it} \int_0^1 \left[ \pi_i(t,a) \left(-\bar{V}_{y_i}^i(t,a,p_{\varepsilon,i}(t,a))\right) - \frac{\beta_{u,i}}{2} \bar{u}_{\varepsilon,i}^2(t,a) \right] dt \\
- \bar{y}_{\varepsilon,i}^0 \int_0^T e^{-r_it} \int_0^1 \left[ \pi_i(t,a) \left(-\bar{V}_{y_i}^i(t,a,\bar{p}_{\varepsilon,i}(t,a))\right) - \frac{\beta_{u,i}}{2} \bar{u}_{\varepsilon,i}^2(t,a) \right] dt \\
- \bar{y}_{\varepsilon,i}^0 \int_0^T e^{-r_it} \left( \frac{\beta_{v,i}}{2} \bar{v}_{\varepsilon,i}^2(t) - \frac{\beta_{u,i}}{2} \bar{u}_{\varepsilon,i}^2(t) \right) dt \\
\geq \bar{y}_{\varepsilon,i}^0 \int_0^1 \bar{V}_{y_i}^i(T,a,p_{\varepsilon,i}(T,a)) da + \bar{y}_{\varepsilon,i}^0 \int_0^T \bar{V}_{y_i}^i(t,1,p_{\varepsilon,i}(t,1))) dt + \varepsilon \bar{y}_{\varepsilon,i}^0 T \\
\geq 3\varepsilon \bar{y}_{\varepsilon,i}^0 T.
\]

The above inequalities follow from the observations: First, using the same arguments as in the proof of (18) and (19), we recognize that

\[
\bar{y}_{\varepsilon,i}^0 \int_0^T e^{-r_it} \int_0^1 \left[ \pi_i(a) \left(-V_{y_i}^i(t,a,p_{\varepsilon,i}(t,a))\right) - \frac{\beta_{u,i}}{2} \bar{u}_{\varepsilon,i}^2(t,a) \right] dt \\
\geq \bar{y}_{\varepsilon,i}^0 \int_0^1 \bar{V}_{y_i}^i(T,a,p_{\varepsilon,i}(T,a)) da + \bar{y}_{\varepsilon,i}^0 \int_0^T \bar{V}_{y_i}^i(t,1,p_{\varepsilon,i}(t,1))) dt + 2\varepsilon \bar{y}_{\varepsilon,i}^0 T
\]

and

\[
- \bar{y}_{\varepsilon,i}^0 \int_0^T e^{-r_it} \int_0^1 \left[ \pi_i(a) \left(-V_{y_i}^i(t,a,\bar{p}_{\varepsilon,i}(t,a))\right) - \frac{\beta_{u,i}}{2} \bar{u}_{\varepsilon,i}^2(t,a) \right] dt \\
\geq -\bar{y}_{\varepsilon,i}^0 \int_0^1 \bar{V}_{y_i}^i(T,a,\bar{p}_{\varepsilon,i}(T,a)) da - \bar{y}_{\varepsilon,i}^0 \int_0^T \bar{V}_{y_i}^i(t,1,\bar{p}_{\varepsilon,i}(t,1))) dt - \varepsilon \bar{y}_{\varepsilon,i}^0 T.
\]

Moreover, similar consideration as in (23) and (24) are applied to show that

\[
-\bar{y}_{\varepsilon,i}^0 \int_0^T \bar{V}_{y_i}^i(t,0,p_{\varepsilon,i}(t,0))) dt \geq \bar{y}_{\varepsilon,i}^0 T + \bar{y}_{\varepsilon,i}^0 \frac{\beta_{v,i}}{2} \int_0^T e^{-r_it} \bar{v}_{\varepsilon,i}^2(t) dt
\]

and

\[
\bar{y}_{\varepsilon,i}^0 \int_0^T \bar{V}_{y_i}^i(t,0,\bar{p}_{\varepsilon,i}(t,0))) dt \geq -\bar{y}_{\varepsilon,i}^0 \frac{\beta_{v,i}}{2} \int_0^T e^{-r_it} \bar{v}_{\varepsilon,i}^2(t) dt
\]

and the assertion follows.
4.2. Numerical algorithm

The above verification theorem allows us to build a numerical approach to calculate the $\varepsilon$-open loop Nash equilibrium in the finite number of steps.

4.2.1. Algorithm scheme

Fix $m_i > 0$ and $\varepsilon > 0$.

Step 1: Define the subset of $\tilde{Ad}_{x_i,\varepsilon}$ consisting of finite elements; i.e., $\tilde{Ad}_{x_i,\varepsilon} \subset Ad_{x_i,\varepsilon}$ (see Section 4.2.2) in the following steps:

Step 1.1 generate the finite number of admissible controls and they belong to $\tilde{U}_{\varepsilon,i} \subset U_{\varepsilon,i}$,

Step 1.2 for player $i$ and each element of the set $\tilde{U}_{\varepsilon,i}$ solve equations (33)-(35).

Step 2: Compute the value of the goal functional $J_i$ for each elements of the set $\tilde{Ad}_{x_i,\varepsilon}$.

Step 3: For player $i$ choose the elements of the set $\tilde{Ad}_{x_i,\varepsilon}$ for which $J_i$ obtains the maximal value and denote them as $(\hat{u}_{\varepsilon,i}, \hat{v}_{\varepsilon,i}, \hat{v}_{\varepsilon,3-i}, \hat{x}_i)$.

Step 4: For player $i$ solve 3D PDE (29)-(32) to obtain $\tilde{V}_i$.

Step 5: Assume $\tilde{y}_{\varepsilon,i} = -1$ and determine $\hat{y}_{\varepsilon,i}(\cdot)$ from the relation
\[
\hat{x}_{\varepsilon,i}(t,a) = -\tilde{V}_i(y_i(t,a_i), \varepsilon_i, t, a) - 1, \hat{y}_{\varepsilon,i}(t,a). \tag{40}
\]

Step 6: For $\tilde{V}_i$ and $(\hat{u}_{\varepsilon,i}, \hat{v}_{\varepsilon,i}, \hat{v}_{\varepsilon,3-i}, \hat{y}_{\varepsilon,i})$ check the inequalities (38)-(39); i.e.,

Step 6.1 if $\tilde{V}_i$ and $(\hat{u}_{\varepsilon,i}, \hat{v}_{\varepsilon,i}, \hat{v}_{\varepsilon,3-i}, \hat{y}_{\varepsilon,i})$ satisfy (38)-(39), then
\[
(\hat{u}_{\varepsilon,i}, \hat{v}_{\varepsilon,i}, \hat{v}_{\varepsilon,3-i})
\]
is an $\varepsilon$-open loop Nash equilibrium and $J_i(\hat{u}_{\varepsilon,i}, \hat{v}_{\varepsilon,i}, \hat{v}_{\varepsilon,3-i})$ is an $\varepsilon$-optimal value,

Step 6.2 otherwise go to Step 1.

In this scheme, one may need to solve 2D and 3D PDE. In our application of this algorithm we use the method of lines in order to transform the PDEs into the systems of ODE. In the case of 2D PDE (i.e. (33)-(35)) we follow a scheme proposed in Górajski and Machowska (2018a). For PDE (29) - (32) in three dimensions we also use the appropriate MOL scheme described in Schiesser and Griffiths (2009, p. 261). In the next section we propose the construction method of the set $\tilde{Ad}_{x_i,\varepsilon}$.
4.2.2. The construction of \( \tilde{Ad}_{x_i, \varepsilon} \)

Based on the observation made in existing literature about goodwill dynamics on a segmented market and on the properties of \( \tilde{Ad}_{x_i, \varepsilon} \) we assume that the dual trajectory takes the form

\[
\bar{y}_{\varepsilon,i}^n(t, a) = e^{-r_it}(1 - a) \left( e^{-c_{i,n}t} - e^{-c_{i,n}T} \right) \left( 1 + R_i(a) \right), \quad (t, a) \in clQ
\]

for

\[
c_{i,n} = 0.1 + 0.01 \cdot n, n \in N := \{1, 2, \ldots, 199\}.
\]

(42)

Based on the dual Hamilton-Jacobi inequality in \( P_i \) given by (29), we propose that the admissible defensive advertising takes the form:

\[
u_{i,n}(t, a) = \left( -\frac{1}{\bar{y}_{\varepsilon,i}^n(t, a)} \bar{y}_{\varepsilon,i}^n(t, a) \frac{\lambda_{u,i}}{\beta_{u,i}} e^{r_it}, \quad (t, a) \in clQ, n \in N, i = 1, 2
\]

(43)

for \( \bar{y}_{\varepsilon,i}(t, a) \) given by (41). The admissible offensive advertising takes the form

\[
u_{i,n}(t) = \left( -\frac{1}{\bar{y}_{\varepsilon,i}^n(t, 0)} \bar{y}_{\varepsilon,i}^n(t, 0) \frac{\lambda_{v,i}}{\beta_{v,i}} e^{r_it}, \quad t \in [0, T], n \in N, i = 1, 2.
\]

(44)

Thus, the admissible controls related to defensive strategies achieve the maximum intensity among customers with little experience in using the product and at the beginning of the decision horizon (similar to Grosset and Viscolani (2016); Górajski and Machowska (2017)). Moreover, the customer recommendations increase the optimal advertising efforts (see Górajski and Machowska (2018b)).

In order to obtain admissible trajectories we solve equation (33)-(35) for each \( (u_{i,n}, v_{i,n}, v_{3-i,n}) \) defined by (43) and (44).

5. Numerical Examples and Sensitivity Analysis

The proposed numerical schema can be employed to analyse how \( \varepsilon \)-open-loop equilibria influence the goodwill path and total profit. We focus on the new factors that have never been studied in the literature related to the goodwill models that describe the segmented oligopolistic market. Therefore, we examine how the degree of homogeneity \( \gamma_{i,3-i} \) and the customer recommendations \( R_i(a) \) have an impact on the optimal results.
5.1. Optimal goodwill

In the first part of the analysis, we examine how goodwill reacts to the equilibrium advertising strategies. In many situations, marketers measure the effectiveness of advertising campaign in terms of the growth in the company’s performance in areas such as goodwill. This study allows us to search for confirmation as to whether this measure of advertising effectiveness is best in the context of maximizing a company’s profit.

We observe that advertising equilibria may increase (see the first column in Fig. 1) or remain (see the first column in Fig. 2) at the initial level of goodwill. Thus, we recognize, that the strengthening or supportive strategies that are defined in the case of a monopolistic market in [Górajski and Machowska (2017)] also appear in the oligopolistic market.

Figure 1: Simulation #1 - the example of the strengthening advertising strategy.

Figure 2: Simulation #2 - the example of the supportive advertising strategy.

Moreover, we analyse when, and for which market segments, firms obtain the maximal level of goodwill. In the first example (see Fig. 1), goodwill
associated with the strengthening of advertising obtains its maximal level at
different moments in different segments. We recognize that the maximum
level of goodwill is achieved later among segments with longer usage expe-
rience. We observe that, in the second example (see Fig. 2), the maximal
level of goodwill in each segment is obtained at the beginning of the decision
horizon and decreases with respect to time $t$. We discover that the high
level of goodwill is maintained longer among customers with the long usage
experience than among those with shorter usage experience.

This result confirms that it is not always optimal in the sense of maximiz-
ing profit to increase the level of goodwill. This finding is consistence with
the thesis proposed in Aaker and Carman (1982) that there is a commonly
observed phenomenon called over-advertising.

5.2. The degree of homogeneity among products

A substantial level of the product homogeneity means that the com-
petition is enhanced and, as a result, the significance of the competitor’s
offensive strategy is increased. Thus, $\gamma_{i,3-i} = 0$ means that the two prod-
ucts are completely different and the market becomes monopolistic. However,
for $\gamma_{i,3-i} = 1$, we obtain a strong competitive market. The results of the
percentage changes with respect to results for the monopolistic market (i.e.
$\gamma_{i,3-i} = 0$) are presented in Fig. 3.

![Figure 3: Simulation #3 - the impact of the degree of products’ homogeneity.](image)

As we anticipated, the growth in competition decreases the goodwill level
and, as a result, also declines the profit of both companies. The results con-
firm the notion that the firms gain the most if they operate in a monopolistic
market or the customers treat the products as completely different goods.
We also see that the challenger loses more than the leader. Thus, to main-
tain an advantage in the market, the challenger’s managers should focus on
the similarities between both products and the leader should emphasize the
differences.
5.3. The impact of customer recommendations

Now, we analyse how the level of own and competitor’s customer recommendations influence the total profit and level of goodwill. We consider two extreme cases. The first describes a situation in which the consumer recommendations of the competing product do not affect the shape of the second player’s reputation (i.e., $\gamma_{i,3-i}^R = 0$ for $i = 1, 2$). This can happen when both competing products are perceived to be completely different products or when the arguments used in the competitor’s product evaluation are not relevant to the decisions of potential consumers of the second product. We also study the other extreme: When consumer rewards of a competing product have a significant impact on the decision of potential consumers (i.e. $\gamma_{i,3-i}^R = 1$). Figures 4 and 5 show the percentage change in the total profit in relation to the case without the influence of consumer recommendations on potential customers for products perceived as different (see Fig. 4) and for substitute products (see Fig. 5).

Figure 4: Simulation #4 - the impact of own customer recommendation on the percentage increase in profit relative to the case that does not include customer recommendations.

The results presented in Fig. 4 confirm the well-established intuition that a high level of positive consumer recommendations significantly increase profit.

The results presented in Fig. 5 underline the relativity of consumer recommendations. Namely, in a situation when competing products are perceived as supplementary products, and consumer recommendations of both products are important for potential consumers then the absolute level of recommendation of a given product is not important; what is important is how the level of recommendation compare to the strength of positive recommendations for a competing product. It is only in this case recommendations are perceived by potential consumers and subsequently influence their decisions.
6. Conclusions

In this paper, a partial differential game of goodwill dynamics was presented that incorporates some new realistic features. The competition between two companies operating in the segmented market according to the customer usage experience was investigated. Both companies manage two types of advertising strategies that are designed to increase goodwill: defensive and offensive. We assume that companies compete for potential consumers through offensive advertising which is directed only at this part of the market. The strength of the competition depends on the level of the product homogeneity. On the other hand, the defensive strategies are focused on the maintenance of the existing customers and are tailored to each consumer. Thus, we are considering personalized advertising activities.

For the new game, we proposed the sufficient condition for the existence of the open-loop Nash advertising equilibrium using the dual approach. Moreover, based on the concept of $\varepsilon$-open-loop equilibrium, we propose the use of a numerical scheme to obtain an optimal solution in the finite steps.

The proposed numerical procedure allowed us to investigate how the homogeneity of competing products affects advertising and goodwill. It transpired both companies lose out when consumers perceive competing products to be similar. However, our investigation indicates that the challenger loses more than the leader. Thus, the challenger’s managers should focus on the similarities between both products to remain competitive in the market while the leader should emphasize the differences between the offering to maintain its competitive advantage in the market.

In the numerical examples, we have shown that, in order to maximize
profit, goodwill should not always increase; sometimes, it is sufficient to maintain it at the existing level. This observation is consistent with the idea that many products are over-advertised. In addition, Kotler and Keller (2012) introduced the concept of an optimal market level that, once exceed, undermines companies’ profitability. Therefore, our research continues this trend and demonstrates that, in some cases, it is not profitable to enhance goodwill.

Finally, we examined how customer recommendations influence the total profit of each company. We confirmed that customer recommendations increase profit and growth is higher if the initial level of goodwill is lower. Moreover, we have detected that, for supplementary products, the relative advantage of the strength of the customer recommendation is important. It seems that this is a significant observation because, according to this view managers should seek customer recommendations for their product that are more audible to potential consumers than those for their competitors’ offering.

References


7. Technical analysis of the numerical simulation

The analysis in Section 5 bases on the several simulations, namely we consider the partial differential game (2) - (4) with (5) and in each simulation we change the value of the parameters $\tilde{R}_i$ and $\gamma_{i,3-i}$ from the sets $\tilde{R}_i \in \{0, 0.1, \ldots, 0.9\}$ and $\gamma_{i,3-i} \in \{0.1, \ldots, 1\}$ for $i = 1, 2$. The rest parameters are fixed at $T = 1$, $\delta_i = 0.1$, $x_{1,0} = 2$, $x_{2,0} = 4$, $\lambda_u = \lambda_v = 0.2$, $\beta_u = \beta_v = 0.03$,
\[ r = 0.028, \quad \gamma_{i,3-i} = 0.5 \] to allow for observation of the impact of the most important parameters in this analysis. It should be mention that the same economical conclusions are obtained for other values of the parameters.

In order to obtain optimal solution we apply the numerical algorithm proposed in Section 4.2.1. For each game we take the dual trajectory of the form (41) and the admissible controls defined by (43) and (44), respectively. In all examples, the accuracy of the computation is equal to \( \varepsilon = 10^{-13} \).

**Simulation #1**

In this simulation the customer recommendation rate is equal to \( R_1 = 0.6 \) and \( R_2 = 0.9 \), the product homogeneity rate - \( \gamma = 0.5 \) and the influence of competitor’s customer recommendations is equal to \( \gamma_R = 0 \). The \( \varepsilon \)-open loop Nash equilibrium is obtained for parameter (42) equal to \( c_{1,64} = 0.73 \) and \( c_{2,56} = 0.65 \).

**Simulation #2**

In this simulation the customer recommendation rate is equal to \( R_1 = 0.6 \) and \( R_2 = 0.9 \), the product homogeneity rate - \( \gamma = 0.5 \) and the influence of competitor’s customer recommendations is equal to \( \gamma_R = 1 \). The \( \varepsilon \)-open loop Nash equilibrium is obtained for parameter (42) equal to \( c_{1,108} = 1.17 \) and \( c_{2,76} = 0.88 \).

**Simulation #3**

In this simulation the customer recommendation rate is equal to \( R_i = 0.4 \) for \( i = 1, 2 \), and the influence of competitor’s customer recommendations is equal to \( \gamma_R = 0.5 \). We assume that the product homogeneity rate is changing and \( \gamma \in \{0, 0.1, \ldots, 0.9, 1\} \). The \( \varepsilon \)-open loop Nash equilibrium is obtained for parameter (42) presented in Table 1.

<table>
<thead>
<tr>
<th>( \gamma )</th>
<th>( c_{i,160} )</th>
<th>( c_{i,153} )</th>
<th>( c_{i,141} )</th>
<th>( c_{i,125} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1.69</td>
<td>1.62</td>
<td>1.5</td>
<td>1.34</td>
</tr>
<tr>
<td>0.4</td>
<td>1.15</td>
<td>0.95</td>
<td>0.78</td>
<td>0.64</td>
</tr>
<tr>
<td>0.8</td>
<td>0.52</td>
<td>0.41</td>
<td>0.32</td>
<td></td>
</tr>
</tbody>
</table>

Table 1: The value for \( c_{i,n} \) for \( \varepsilon \)-open loop Nash equilibrium in simulation #3.
Simulation #4

In this simulation we assume that the product homogeneity rate is equal to $\gamma = 0.5$ and the influence of competitor’s customer recommendations is equal to $\gamma^R = 0$. The $\varepsilon$-open loop Nash equilibrium is obtain for parameter (42) presented in Table 2.

Simulation #5

In this simulation we assume that the product homogeneity rate is equal to $\gamma = 0.5$ and the influence of competitor’s customer recommendations is equal to $\gamma^R = 1$. The $\varepsilon$-open -loop Nash equilibrium is obtain for parameter (42) presented in Table 3.
<table>
<thead>
<tr>
<th>$R_2$</th>
<th>$R_1 = 0$</th>
<th>$R_1 = 0.1$</th>
<th>$R_1 = 0.2$</th>
<th>$R_1 = 0.3$</th>
<th>$R_1 = 0.4$</th>
<th>$R_1 = 0.5$</th>
<th>$R_1 = 0.6$</th>
<th>$R_1 = 0.7$</th>
<th>$R_1 = 0.8$</th>
<th>$R_1 = 0.9$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>$c_{1,0}$ = 1.17</td>
<td>$c_{1,0.1}$ = 1.05</td>
<td>$c_{1,0.2}$ = 0.95</td>
<td>$c_{1,0.3}$ = 0.88</td>
<td>$c_{1,0.4}$ = 0.77</td>
<td>$c_{1,0.5}$ = 0.68</td>
<td>$c_{1,0.6}$ = 0.59</td>
<td>$c_{1,0.7}$ = 0.68</td>
<td>$c_{1,0.8}$ = 0.65</td>
<td>$c_{1,0.9}$ = 0.65</td>
</tr>
<tr>
<td>0.1</td>
<td>$c_{2,0}$ = 1.17</td>
<td>$c_{2,0.1}$ = 1.05</td>
<td>$c_{2,0.2}$ = 0.95</td>
<td>$c_{2,0.3}$ = 0.88</td>
<td>$c_{2,0.4}$ = 0.77</td>
<td>$c_{2,0.5}$ = 0.68</td>
<td>$c_{2,0.6}$ = 0.59</td>
<td>$c_{2,0.7}$ = 0.68</td>
<td>$c_{2,0.8}$ = 0.65</td>
<td>$c_{2,0.9}$ = 0.65</td>
</tr>
<tr>
<td>0.2</td>
<td>$c_{3,0}$ = 1.17</td>
<td>$c_{3,0.1}$ = 1.05</td>
<td>$c_{3,0.2}$ = 0.95</td>
<td>$c_{3,0.3}$ = 0.88</td>
<td>$c_{3,0.4}$ = 0.77</td>
<td>$c_{3,0.5}$ = 0.68</td>
<td>$c_{3,0.6}$ = 0.59</td>
<td>$c_{3,0.7}$ = 0.68</td>
<td>$c_{3,0.8}$ = 0.65</td>
<td>$c_{3,0.9}$ = 0.65</td>
</tr>
<tr>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
</tr>
<tr>
<td>0.9</td>
<td>$c_{4,0}$ = 1.17</td>
<td>$c_{4,0.1}$ = 1.05</td>
<td>$c_{4,0.2}$ = 0.95</td>
<td>$c_{4,0.3}$ = 0.88</td>
<td>$c_{4,0.4}$ = 0.77</td>
<td>$c_{4,0.5}$ = 0.68</td>
<td>$c_{4,0.6}$ = 0.59</td>
<td>$c_{4,0.7}$ = 0.68</td>
<td>$c_{4,0.8}$ = 0.65</td>
<td>$c_{4,0.9}$ = 0.65</td>
</tr>
</tbody>
</table>

Table 2: The value for $c_{i,n}$ for $\varepsilon$-open loop Nash equilibrium in simulation #4.
<table>
<thead>
<tr>
<th>$R_1$</th>
<th>$R_1 = 0$</th>
<th>$R_1 = 0.1$</th>
<th>$R_1 = 0.2$</th>
<th>$R_1 = 0.3$</th>
<th>$R_1 = 0.4$</th>
<th>$R_1 = 0.5$</th>
<th>$R_1 = 0.6$</th>
<th>$R_1 = 0.7$</th>
<th>$R_1 = 0.8$</th>
<th>$R_1 = 0.9$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$R_2 = 0$</td>
<td>$c_{1,108} = 1.17$</td>
<td>$c_{1,96} = 1.05$</td>
<td>$c_{1,86} = 0.95$</td>
<td>$c_{1,79} = 0.88$</td>
<td>$c_{1,73} = 0.82$</td>
<td>$c_{1,68} = 0.77$</td>
<td>$c_{1,64} = 0.73$</td>
<td>$c_{1,61} = 0.70$</td>
<td>$c_{1,59} = 0.68$</td>
<td>$c_{1,56} = 0.65$</td>
</tr>
<tr>
<td>$R_2 = 0.1$</td>
<td>$c_{1,108} = 1.17$</td>
<td>$c_{1,108} = 1.17$</td>
<td>$c_{1,96} = 1.05$</td>
<td>$c_{1,86} = 0.95$</td>
<td>$c_{1,79} = 0.88$</td>
<td>$c_{1,73} = 0.82$</td>
<td>$c_{1,68} = 0.77$</td>
<td>$c_{1,64} = 0.73$</td>
<td>$c_{1,61} = 0.70$</td>
<td>$c_{1,59} = 0.68$</td>
</tr>
<tr>
<td>$R_2 = 0.2$</td>
<td>$c_{1,108} = 1.17$</td>
<td>$c_{1,108} = 1.17$</td>
<td>$c_{1,96} = 1.05$</td>
<td>$c_{1,86} = 0.95$</td>
<td>$c_{1,79} = 0.88$</td>
<td>$c_{1,73} = 0.82$</td>
<td>$c_{1,68} = 0.77$</td>
<td>$c_{1,64} = 0.73$</td>
<td>$c_{1,61} = 0.70$</td>
<td>$c_{1,59} = 0.68$</td>
</tr>
<tr>
<td>$R_2 = 0.3$</td>
<td>$c_{1,108} = 1.17$</td>
<td>$c_{1,108} = 1.17$</td>
<td>$c_{1,96} = 1.05$</td>
<td>$c_{1,86} = 0.95$</td>
<td>$c_{1,79} = 0.88$</td>
<td>$c_{1,73} = 0.82$</td>
<td>$c_{1,68} = 0.77$</td>
<td>$c_{1,64} = 0.73$</td>
<td>$c_{1,61} = 0.70$</td>
<td>$c_{1,59} = 0.68$</td>
</tr>
<tr>
<td>$R_2 = 0.4$</td>
<td>$c_{1,108} = 1.17$</td>
<td>$c_{1,108} = 1.17$</td>
<td>$c_{1,96} = 1.05$</td>
<td>$c_{1,86} = 0.95$</td>
<td>$c_{1,79} = 0.88$</td>
<td>$c_{1,73} = 0.82$</td>
<td>$c_{1,68} = 0.77$</td>
<td>$c_{1,64} = 0.73$</td>
<td>$c_{1,61} = 0.70$</td>
<td>$c_{1,59} = 0.68$</td>
</tr>
<tr>
<td>$R_2 = 0.5$</td>
<td>$c_{1,108} = 1.17$</td>
<td>$c_{1,108} = 1.17$</td>
<td>$c_{1,96} = 1.05$</td>
<td>$c_{1,86} = 0.95$</td>
<td>$c_{1,79} = 0.88$</td>
<td>$c_{1,73} = 0.82$</td>
<td>$c_{1,68} = 0.77$</td>
<td>$c_{1,64} = 0.73$</td>
<td>$c_{1,61} = 0.70$</td>
<td>$c_{1,59} = 0.68$</td>
</tr>
<tr>
<td>$R_2 = 0.6$</td>
<td>$c_{1,108} = 1.17$</td>
<td>$c_{1,108} = 1.17$</td>
<td>$c_{1,96} = 1.05$</td>
<td>$c_{1,86} = 0.95$</td>
<td>$c_{1,79} = 0.88$</td>
<td>$c_{1,73} = 0.82$</td>
<td>$c_{1,68} = 0.77$</td>
<td>$c_{1,64} = 0.73$</td>
<td>$c_{1,61} = 0.70$</td>
<td>$c_{1,59} = 0.68$</td>
</tr>
<tr>
<td>$R_2 = 0.7$</td>
<td>$c_{1,108} = 1.17$</td>
<td>$c_{1,108} = 1.17$</td>
<td>$c_{1,96} = 1.05$</td>
<td>$c_{1,86} = 0.95$</td>
<td>$c_{1,79} = 0.88$</td>
<td>$c_{1,73} = 0.82$</td>
<td>$c_{1,68} = 0.77$</td>
<td>$c_{1,64} = 0.73$</td>
<td>$c_{1,61} = 0.70$</td>
<td>$c_{1,59} = 0.68$</td>
</tr>
<tr>
<td>$R_2 = 0.8$</td>
<td>$c_{1,108} = 1.17$</td>
<td>$c_{1,108} = 1.17$</td>
<td>$c_{1,96} = 1.05$</td>
<td>$c_{1,86} = 0.95$</td>
<td>$c_{1,79} = 0.88$</td>
<td>$c_{1,73} = 0.82$</td>
<td>$c_{1,68} = 0.77$</td>
<td>$c_{1,64} = 0.73$</td>
<td>$c_{1,61} = 0.70$</td>
<td>$c_{1,59} = 0.68$</td>
</tr>
</tbody>
</table>

Table 3: The value for $c_{i,n}$ for ε-open loop Nash equilibrium in simulation #5.